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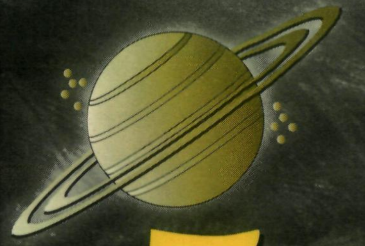
500 Years of Tantrasangraha A Landmark in the History of Astronomy

M. S. Sriram
K. Ramasubramanian
M. D. Srinivas

IIAS



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The year 2000 was the five-hundredth anniversary of the composition of the celebrated astronomical text *Tantrasangraha* by the renowned Kerala astronomer Nilakantha Somayaji (c. 1444-1545 AD) of Trikkantiyur. *Tantrasangraha* ranks along with *Aryabhatiya* (c. 499 AD) of Aryabhata and *Siddhantasiromani* (c. 1150 AD) of Bhaskaracharya as one of the major works which significantly influenced all further work on Astronomy in India.

In *Tantrasangraha,* Nilakantha introduced a major revision of the traditional Indian planetary model. He arrived at a unified theory of planetary latitudes and a better formulation of the equation of centre for the interior planets (mercury and Venus) than was available, either in the earlier Indian works, or in the Islamic or European traditions of Astronomy till the work of Kepler. In his other works *Golasara, Siddhantadarpana* and *Aryabhatiyabhashya*, Nilakantha outlined the geometrical picture of planetary motion that follows from his model. According to this picture, the five planets Mercury, Venus, Mars, Jupiter and Saturn go around the Sun which in turn goes around the Earth.

During 11-13 March, 2000, the Department of Theoretical Physics, University of Madras, organised a Conference to celebrate the 500th Anniversary of *Tantrasangraha*, in collaboration with the Inter-University Centre of the Indian Institute of Advanced Study, Shimla. The Conference turned out to be an important occasion for highlighting and reviewing the recent work on the avhievements in Mathematics and Astronomy of the Kerala school and the new perspectives in History of Science, which are emerging from these studies. This volume is a compilation of the important papers presented at this Conference.

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500 YEARS OF TANTRASANGRAHA

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K. RAMASUBRAMANIAN
M.D. SRINIVAS

INTER-UNIVERSITY CENTRE
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PREFACE

The year 2000 AD happens to be the five-hundredth anniversary of the composition of the celebrated astronomical text *Tantrasangraha* by the renowned Kerala astronomer Nilakantha Somayaji (c. 1444 - 1545 AD) of Trikkantiyur. *Tantrasangraha* ranks along with *Aryabhatiya* (c. 499 AD) of Aryabhata and *Siddhantasiromani* (c. 1150 AD) of Bhaskaracharya as one of the major works which significantly influenced all further work on Astronomy in India.

The Kerala School of *Jyotisha*, starting with Madhava of Sangamagrama (c. 1340-1425 AD), is well-known for its pioneering work on mathematical analysis, especially the discovery of infinite series for π, sine and cosine functions and the development of fast convergent approximations for them. Their significant contributions in Astronomy, especially in planetary theory, the computation of eclipses and spherical astronomy, are being highlighted only by some recent studies.

In *Tantrasangraha*, Nilakantha Somayaji introduced a major revision of the traditional Indian planetary model. He arrived at a unified theory of planetary latitudes and a better formulation of the equation of centre for the interior planets (Mercury and Venus) than was available, either in the earlier Indian works, or in the Islamic or European traditions of Astronomy till the work of Kepler, which was to come more than a hundred years later. Thus the composition of *Tantrasangraha* in 1500 AD is indeed a major landmark in the History of Astronomy.

In his later works, *Golasara*, *Siddhantadarpana* and the famous *Aryabhatiyabhashya*, Nilakantha also discussed the geometrical picture of planetary motion, implied by his computational scheme, according to which the five planets, Mercury, Venus, Mars, Jupiter and Saturn go in eccentric orbits around the mean Sun, which in turn goes around the Earth.

Apart from introducing the improved planetary model, *Tantrasangraha* also presents a systematic exposition of all aspects of the Indian tradition of mathematical astronomy in about 432 verses. It includes several novel and more accurate results on the computation of instantaneous velocities of planets (including the well-known result for the derivative of the inverse-sine function), of eclipses, and a host of other quantities in spherical astronomy. It is clearly the most important treatise on Astronomy composed by the Kerala School of Astronomers.

Sankara Variyar, in his commentary *Laghuvivritti* (c. 1556 AD) on

Tantrasangraha, notes that the text contains two chronograms, one at the beginning and one at the end, indicating the time of its composition. This shows that the text was begun when the *Kali ahargana* (number of civil days elapsed since the beginning of *Kaliyuga,* viz. February 18, 3102 BC) was 16,80,548 (March 22, 1500 AD, as per the Gregorian Calendar) and completed when the *Kali ahargana* was 16,80,553 (March 27, 1500 AD). Thus 2000 AD happens to be the 500th Anniversary of this great landmark in History of Astronomy.

During 11th-13th March 2000, the Department of Theoretical Physics, University of Madras organised a Conference, to celebrate the 500th Anniversary of *Tantrasangraha.* The Conference was organised in collaboration with the Inter-university Centre for Humanities and Social Sciences of the Indian Institute of Advanced Study, Shimla. The Conference turned out to be an important occasion for highlighting and reviewing the recent work on the achievements in Mathematics and Astronomy of the Kerala School, and the new perspectives in History of Science, which are emerging from these studies. This volume happens to be a compilation of the important papers presented at this Conference.

The renowned scholar of Indian tradition of Astronomy and Mathematics, Prof. K. V. Sarma, delivered the first lecture of the Conference on Nilakantha and his works. In his contribution, Prof. Sarma has given an overview of what is known about the life of Nilakantha. He has also presented a detailed account of the contents and important results found in Nilakantha's works: *Tantrasangraha, Golasara, Siddhanta-darpana* and its *Vyakhya, Chandracchayaganita* and its *Vyakhya, Grahaparikshakrama, Sundararajaprasnottara, Aryabhatiyabhashya* and *Jyotirmimamsa.*

Prof. M. S. Sriram of Madras University has presented an overview of the development of planetary theory in the Greeko-European and Indian astronomical traditions. The traditional Indian planetary model, as given in the works of Aryabhata and other ancient astronomers, worked quite well for the exterior planets (Mars, Jupiter and Saturn) because the *manda* correction corresponds to the equation of centre and the *sighra* correction converts the heliocentric longitudes to geocentric longitudes. The traditional model, however, incorrectly applied the equation of centre for the interior planets (Mercury and Venus) to the mean Sun. Still, the Indian astronomers managed to give a fairly accurate procedure for computing the latitudes of the planets, even for the case of the interior planets, by prescribing the use of the so-called *sighroccha* for computing the latitudes.

The success of the ancient Indian astronomers was in marked contrast with the Greek planetary model of Ptolemy, where the planes of all the planetary orbits intersected at the Earth and so it became almost impossible to capture even the basics of the latitudinal motions of planets. The problems concerning the planetary latitudes and the equation of centre for interior planets persisted with the celebrated reformulations of Copernicus and Tycho Brahe (as they followed Ptolemy uncritically in most respects) and was resolved in the Greeko-European tradition only with the work of Kepler in early 17th century.

Dr. K. Ramasubramanian of Madras University has discussed Nilakantha's revision of the traditional Indian planetary model as presented in *Tantrasangraha* and further elaborated in his *Aryabhatiyabhashya*. Nilakantha unified the two seemingly different procedures of computing planetary latitudes in the traditional planetary theory. He proposed that what was taken as the *sighroccha* of the interior planet in the traditional model should be identified with the planet itself – as what was observed as the latitude was the deflection of the planet from the ecliptic and not of some *sighroccha*. Nilakantha also proposed that the *manda* correction or the equation of centre for the interior planet should be applied to the mean planet (referred to as *sighroccha* by the ancients) and not to the mean Sun.

In this way Nilakantha arrived at the correct formulation of the equation of centre for the interior planets; he seems to have been the first astronomer to do so in the history of astronomy. Nilakantha also gave a unified theory of planetary latitudes. This revision of the traditional planetary model as given in *Tantrasangraha* seems to have been accepted by most of the later astronomers of Kerala such as Chitrabhanu, Sankara Variyar, Jyeshthadeva, Achyuta Pisharati and Putumana Somayaji.

Prof. M. D. Srinivas of Centre for Policy Studies, Madras, has outlined the geometrical picture of planetary motion according to Nilakantha. Unlike in the Greeko-European tradition, which presented exclusively geometrical models for planetary motion (and that too based solely on combinations of uniform circular motions), the Indian astronomers by and large present computational/analytical models and occasionally discuss geometrical pictures as aids to understanding. One such discussion of the geometrical picture associated with the traditional planetary model may be found in the commentary *Bhatadipika* on *Aryabhatiya* by Paramesvara, the *Paramaguru* of Nilakantha.

In his *Aryabhatiyabhashya*, Nilakantha notes that the geometrical picture of planetary motion depends on the computational model

employed. In the same work Nilakantha clearly explains that the interior planets go around the Sun in orbits that do not enclose the Earth and that they go around the Earth only by virtue of the fact that the Sun goes around the Earth. In his works *Golasara* and *Siddhantadarpana*, Nilakantha presents a concise but precise statement of the geometrical picture of planetary motion. According to this picture, the five planets, Mercury, Venus, Mars, Jupiter and Saturn, go around the mean Sun in eccentric orbits; the planes of the orbits of the planets are inclined to the ecliptic and pass through the mean Sun.

The geometrical picture associated with the traditional planetary model of Indian astronomy also turns out to be fairly interesting, once some of its special features, such as the variable nature of the epicycles, are also taken into account. In his contribution, Prof. S. Madhavan of Thiruvananthapuram has explained how the variable epicycle model given in the *Suryasiddhanta* for the *manda* process, leads to the orbit of the planet being made up of two elliptical segments, which are asymmetrical.

The flexible theoretical framework of Indian astronomy is highlighted in the contribution of Prof. S. Balachandra Rao of National College, Bangalore, and his co-workers Smt. Padmaja Venugopal and Smt. S. K. Uma. Making use of the variable epicycle model of the Aryabhata School, they have introduced suitable *bija* corrections for the peripheries of the epicycles while computing the true positions of planets, so that the results turn out to be fairly accurate for contemporary times.

Prof. L. Satpathy of the Institute of Physics, Bhubaneswar, has discussed the work of the great Astronomer of Orissa, Chandrasekhara Samanta, who flourished in the 19th century. Working under trying circumstances and without any training in modern astronomy, Chandrasekhara arrived at a very similar geometrical picture of planetary motion (as discussed by Nilakantha) and was also able to obtain all the four important corrections for the lunar motion, the *manda* correction (equation of centre), *tungantara* (which incorporates evection), *pakshika* (variation) and *digamsa* (annual equation). It was no wonder that the council of the Puri temple decided to follow the *panchanga* as computed by Chandrasekhara, as it was in best accordance with observations.

A couple of papers presented in the Conference dealt with the important issue of interactions between the Indian astronomical tradition and other traditions in astronomy. In his contribution, Prof. S. M. R. Ansari of Aligarh has discussed the development of Islamic astronomical tradition in India and its interaction with traditional Indian astronomy. He has focussed on the various Observatories that were built and the Astronomical

Tables (*Zij*) that were prepared in India during 16th - 18th centuries. He has also discussed the impact of Islamic astronomical tradition and the debates that it gave rise to in the works of Nityananda, Kamalakara and Muniswara in the 16th and 17th centuries and also in the works of the astronomers in the court of Raja Sawai Jai Singh in the 18th century.

In their contribution, the young historians of Astronomy from Iran, Dr. Farid Ghassemlou and Dr. Negar Naderi, have discussed the Persian Astronomical Tables that were prepared in India. Outlining their intensive search for such Persian Astronomical Tables in India, Iran and elsewhere, they have summarised the contents of 11 Astronomical Tables and their relation with Indian Astronomy.

Any Conference on the work of the Kerala School would be incomplete without some discussion on the significant contributions made by the Kerala School to the development of mathematical analysis. Dr. Madhukar Mallaya of Thiruvananthapuram has discussed the geometrical demonstrations given in Nilakantha's *Aryabhatiyabhashya* for the well-known results of Aryabhata concerning the sums of natural numbers, sums of their squares and sums of their cubes. This elegant demonstration involving rectangular strips is indeed of great pedagogical value.

It is well known that the series for $\pi/4$ obtained by Madhava in the fourteenth century (the so-called Gregory series) is an inordinately slowly convergent series. Madhava also prescribed two correction or remainder terms, which give rise to better approximations. Dr. Jolly John of Nehru Memorial Library, New Delhi, has presented the rationale given in the celebrated Malayalam text *Yuktibhasha* of Jyeshthadeva (c. 16th century) for obtaining the correction terms which enable us to easily compute values of π correct to any level of accuracy from this slowly converging series. He has also explained how the rationale given in *Yuktibhasha* can be used to derive higher order correction terms, which essentially lead to a well-known continued fraction.

The Conference also honoured Prof. K. V. Sarma for his extraordinary contributions to the study of Indian Mathematics and Astronomy. The Sanskrit citation prepared for this occasion by Dr. K. Ramasubramanian paid glowing tributes to Prof. Sarma for his monumental work of identifying, editing and publishing a large number of source-works in Indian Mathematics and Astronomy, especially the classics of the Kerala School. His work has significantly contributed to dispel many widely prevalent myths, that the Indian tradition of Mathematics and Astronomy ended with Bhaskaracharya II (c. 1150 AD), that the notion of logical rigour and proof was absent in this tradition, and so on. His publication

of seminal works such as the *Jyotirmimamsa* of Nilakantha has brought to light the philosophical approach so characteristic of Indian scientists that scientific theories need constant updating and revision in the light of continuing observations. We have included the Sanskrit citation presented to Prof. K. V. Sarma in this volume.

We would like to express our gratitude to the contributors to this volume and the other participants in the Conference who made it a memorable event. We are especially indebted to Prof. V. C. Srivastava, the Director of the Indian Institute of Advanced Study, Shimla, and his colleagues for enthusiastically supporting this Conference and for extending all co-operation.

M. S. SRIRAM
K. RAMASUBRAMANIAN
M. D. SRINIVAS

Nilakantha Somayaji (1444-1545 AD)
The astute astronomer of Kerala and his works

K.V. Sarma

Nilakantha Somayaji

Nilakantha Somayaji, the astute astronomer of Kerala and author of several works on Indian astronomy, shines as one of the brightest stars in the firmament of Indian astronomy. A versatile scholar, Nilakantha initiated several new and accurate methods of astronomical computations relating to the instantaneous velocities of the planets, the eclipses, measurement of time from the Sun's and Moon's shadows and a host of other matters. To his credit go the depiction of a simpler and more accurate planetary model of the interior planets and the computation of the equation of centre. The novel theories envisaged in his works form a pointer to the great strides that the development of the astronomical and mathematical studies had taken in Kerala during medieval times.

Autobiographical details

Our author Nilakantha is generally referred to with the title *Somayaji*, *Somasut*, *Somasutvan* or *Comatiri*, the last being the Malayalam derivative of the Sanskrit words. A detailed colophon occurring at the end of his *Bhashya* on the *Ganitapada* of the *Aryabhatiya*, contains a good deal of information about him[1]:

इति श्री-कुण्डग्रामजेन गार्ग्यगोत्रेण आश्वलायनेन भाट्टेन केरलसद्ग्रामगृहस्थेन श्रीश्वेतारण्यनाथपरमेश्वरकरुणाधिकरणभूतविग्रहेण जातवेद:पुत्रेण शंकराग्रजेन जातवेदोमातुलेन दृग्गणितनिर्मापकपरमेश्वरपुत्र-श्रीदामोदरात्तज्योतिषामयनेन रवित आत्तवेदान्तशास्त्रेण सुब्रह्मण्यसहृदयेन नीलकण्ठेन सोमसुता विरचितविविधग्रन्थेन दृष्टबहूपपत्तिना स्थापितपरमार्थेन कालेन शंकराय निर्मिते श्रीमदार्यभटसिद्धान्तव्याख्याने महाभाष्ये. . .

[1] *Aryabhatiya*, Trivandrum Sanskrit Series (TSS), No. 101 (Trivandrum, 1930), p. 180.

The above-quoted passage informs that Nilakantha belonged to the *Gargya-gotra*[2], was a follower of the *Asvalayana-sutra* of the *Rigveda* and was a *Bhatta*. He was the son of Jatavedas and had a younger brother named Sankara. He had an Uncle Jatavedas by name and a close friend Subramanya. He was a perfomer of the *Soma* sarifice. He had composed several works on astronomy, in which subject he had made deep and extensive investigations, a fact which is well borne out by his available works.

Some more personal details about Nilakantha seem to be forthcoming from a Malayalam work entitled *Laghuramayanam*. This work describes itself as a work of Rama, son of Nilakantha of the *Gargyagotra* and resident of Kundagrama. Cf., the colophon at its end[3]:

इति कुण्डग्रामजेन गार्ग्यकुलतिलकेन श्रीनीलकण्ठात्मजेन आर्याम्बागर्भसम्भवेन मन्वादिस्मृतिमर्मज्ञ-संस्कृत-द्राविड-भाषा-त्रयपारीणस्य दक्षिणामूर्तिनाम्नोऽग्रजेन रामेण विरचितं श्रीरामायणं प्रबन्धम्।

This Nilakantha is identified by the editor of the work with our author[4]. If this identification is correct, Nilakantha's wife was named Arya, and he had two sons Rama and Dakshinamurti, the latter of whom was well versed in the *Dharmasastras* and learned in the three languages, Sanskrit, Tamil and Malayalam. The great Malayalam poet Tuncattu Ezhuttacchan is said to have been a student of Nilakantha. Nilakantha is also said to have composed, at the request of a friend, a panegyric in Malayalam on the Goddess Parvati, the presiding deity of the temple of Urakam in Cochin, in order to ward off the predicted premature death of that friend's daughter[5]. The authenticity of the above work and the sources of the information are however not quite certain, and corroborative evidences have to be found before accepting the above statements.

Native village and favourite deity

Nilakantha hailed from Trik-kanti-yur (sanskritised into Sri-Kunda-pura or Sri-Kunda-grama), near Tirur, S.Rly., Ponnani Taluk, South Malabar,

[2] The term *Gargya* is often affixed to his name in references.

[3] Ed. P.R. Menon, *Tuncattu Granthavali*, No. 3, Tuncattu Karyalayam, Chittoor, 2nd edn., 1931.

[4] Vide P.R. Menon in his article *'Tuncattu Ezhuttacchan'* in the Malayalam monthly, Tuncattu Ezhuttacchan, 3 (1952-53) 127-35.

[5] *Ibid.* This *stotra* is published in a collection of *stotras* in Malayalam script entitled *Stavaratnamala,* Pt. I.

a famous seat of learning in Kerala during the middle ages. The name of his *Illam*, as the house of a Namputiri Brahmin is called, was *Kelallur* (sometimes spelt also as *Kerallur* and *Kalannur*), sanskritised into *Kerala-sad-grama* corresponding to the Malayalam word *Kerala-nall-ur*.[6] Nilakantha's house is identified as the present Etamana Illam, situated a little to the south of the local temple[7]. It is stated that Nilakantha's family became extinct and that the family property was inherited by the nearest relations, *viz*., the Etamana family.

Nilakantha's favourite deity was Lord Siva installed in the famous temple at Tripparannod (Sanskrit, Sri-Parakroda, also Sri-Svetaranya) near his village; cf.: श्वेतारण्यनाथ-परमेश्वर-करुणाधिकरणभूतविग्रहेण, in the colophon to the *Aryabhatiyabhashya* quoted above.

Patron, Netranarayana, and brother Sankara

Nilakantha refers to his younger brother Sankara in several places in the *Aryabhatiyabhashya.* Sankara too seems to have been well versed in astronomy and to have followed his elder brother's studies. Thus, after describing some method on the rule of three *(trairasika)* in his *Aryabhatiyabhashya, Ganita.* 26, Nilakantha says how his brother who was teaching at the house of his patron explained to the latter some of those theories[8]:

अत्र केषाञ्चिद् युक्तयः पुनः अस्मदनुजेन शंकराख्येन तत्समीपे अध्यापयता वर्तमानेन तस्मै (नेत्रनारायणाय) प्रतिपादिताः।

Nilakantha observes at the close of the *Bhashya* on the *Golapada* that he was entrusting the *Bhashya* to Sankara for its proper propagation. Thus, just before the final colophon, Nilakantha says[9]:

एवमिदम् अस्माभिर्यथामति व्याख्यातम्।
नमः स्वयम्भुवे तस्मै यत्प्रसादादिदं कृतम्।
नमो भगवते तस्मै श्रीमदार्यभटाय च।।

[6] It may be noted that in the expression *Gargya-Kerala* prefixed to the author's name, the word *Kerala* refers to the name of his house and not to the state, as is sometimes taken.

[7] Cf., Vatakkumkur Rajaraja Varma, *History of Sans. Lit. in Kerala,* vol. I, Trivandrum, 1938, p.384.

[8] *Aryabhatiya,* TSS No. 101 (Trivandrum, 1930), p. 156.

[9] *Aryabhatiya*, TSS No. 185 (Trivandrum, 1957), p. 156.

शिष्यं तत्त्वेन विचार्यार्यभटसूत्रभाष्यमिदम्।
यदि स न्यायाल्लिप्सेदस्मै दातव्यमेव शंकर ते।।

That Nilakantha was intimately connected with and was patronised by Kausitaki Adhya Netranarayana, known locally as Azhvanceri Tamparakkal, the religious head of the Namputiri Brahmins of Kerala, is known from several references in his writings. It is also clear that the patron had great esteem for Nilakantha's erudition in astronomy, in which subject he too was interested and used to discuss difficult points with Nilakantha. Thus, in the discussion on the calculation of the motion of planets (*Aryabhatiya., Kala.,* 22-25), Nilakantha says[10]:

कर्णभुक्तिः स्फुटेत्यत्र व्याख्याने पारमेश्वरे।
व्यासार्धाप्तं कोटिवर्गात् कर्ण्येणादावृणं धनम्।।
कोट्यां तदूनयुग्न्यासदलं गतिविधौ श्रुतिः।
प्रकारान्तरमाहैवं सूक्ष्मभुक्तिप्रसिद्धये।।
गुरूणां मे पितात्रापि स्थौल्यान्मत्सरिणोदिते।
परमेश्वरतच्छिष्या नैव वेलागतिं विदुः।।
इति कौषीतकी श्रुत्वा नेत्रनारायणः प्रभुः।
मह्यं न्यवेदयत् तस्मै तदेवं प्रत्यपादयम्।।

Again, in the long discussion on the calculation of the apparent position of celestial bodies (*Aryabhatiya., Kala.,* 17-21), speaking on a method to derive the *sakrit-karna,* our author says [11]:

अन्यदपि कर्म अस्माभिरुपन्यस्यमानं श्रुत्वा आढ्येन कौषीतकिना अनुष्टुभा निबद्धम्-
स्वोच्चोनमध्यमार्कस्य भुजाज्याघ्ना त्रिजीविका।
स्वोच्चहीनस्फुटार्कस्य दोर्ज्याभक्ता श्रुतिर्भवेत्।।

This would indicate the intimacy that existed between Nilakantha and his patron and the common interest that bound them together. On the compilation of the *Aryabhatiyabhashya,* Nilakantha observes in one place[12]:

यन्मयात्र केषाञ्चित् सूत्राणां तद्युक्तीः प्रतिपाद्य कौषीतकिना आढ्येन नारायणाख्येन व्याख्यानं कारितम्, अतस्तदेवात्र लिख्यते।

Again, in another context, he remarks[13]:

[10] *Aryabhatiya*, TSS No. 110 (Trivandrum, 1931), p. 63.
[11] *Ibid.*, p. 47.
[12] *Aryabhatiya*, TSS No. 101 (Trivandrum, 1930), p. 113.
[13] *Ibid.*, p. 156.

इतीदं प्रथमे वयस्येव वर्तमानेन मया द्वितीयवयसि स्थितेन कौषीतकिना आढ्येन कारितम्। . . . . तस्मिन् स्वर्गते पुनः . . . . व्याख्यानमारब्धम्।

It is clear from the above that the credit of enthusing Nilakantha in his investigations, and in fact, to have prompted him to write his *Aryabhatiyabhashya,* goes to Netranarayana,[14] the members of whose family are known all through the annals of Kerala history to have been good scholars and, at the same time, liberal patrons of scholarship.

Nilakantha's teachers Ravi and Damodara

Nilakantha informs us in his *Aryabhatiyabhashya* that he studied *Vedanta* under Ravi, cf. *Ravita atta-Vedanta-sastrena*[15]. That Ravi was well versed also in *Jyotisssatra* and that Nilakantha imbibed some of his knowledge in astronomy also from Ravi is clear from the introductory verse to Nilakantha's *Siddhantadarpana,* where his teachers have been mentioned by double entendre:

श्रीमद्दामोदरं नत्वा भगवन्तं रविं तथा।
यत्प्रसादान्मया लब्धं ज्योतिश्चरितमुच्यते।।

A work on astrology, *Acaradipika,* which is a detailed commentary in verse on *Muhurtashtaka,* is ascribed to this Ravi[16].

The teacher of Nilakantha who actually initiated him into the science of astronomy and instructed him on the various principles underlying *mathematical calculations was Damodara, son of the Kerala-Drigganita* author Paramesvara[17], of the *Bhargava-gotra* and resident of the village of Alattur (sanskritised into *Asvattha-grama*) which was situated quite near Nilakantha's own village. In his *Aryabhatiyabhashya,* as also in his other works, Nilakantha reverentially refers to his teacher and his studies under him. He speaks of how even as a boy he stayed with his *Guru*, at the latter's residence, prosecuting his studies[18]:

[14] Even with regard to Nilakantha's *Tantrasangraha*, its introductory verse, हे विष्णो निहितं कृत्स्नं जगत् त्वय्येव कारणे। ज्योतिषां ज्योतिषे तस्मै नमो नारायणाय ते।। has a veiled reference to his patron (Netra)Narayana at whose instance that work too seems to have been written.

[15] *Aryabhatiya*, TSS No. 101 (Trivandrum, 1930), p. 180.

[16] Ulloor, *Kerala Sahitya Caritram*, vol. II, Trivandrum (1954), p.114. For a manuscript of this work see Kerala Uni. Mss. Lib., No. 3336-B.

[17] Cf. the detailed colophon quoted above,

[18] *Aryabhatiya*, TSS, No. 110 (Trivandrum, 1931), p. 48.

मया गुरुकुले वसता बाल्य एव . . . .
निबद्धं च तत् तदैव अस्मद्गुरुभिः पञ्चभिरुपजातिभिः 'अर्कस्फुटेनानयनं प्रकुर्यात्' . . .। तदपि – 'सर्वत्र विष्कम्भदलं श्रुतौ वा व्यासार्धके स्याद् विपरीतकर्णः' इत्यस्मद्गुरुणोक्तम्।[19]

Similar quotations and other references, which Nilakantha and later authors make, proclaim Damodara not only to be a prominent astronomer of the times but also as the author of erudite works on the subject, manuscripts of which are yet to come to light.

Nilakantha followed the footsteps of Paramesvara, founder of the *Drigganita* system of astronomy in Kerala and one of the foremost astronomers of the land. For him Paramesvara was not only the revered father of his *Guru* but was also his *Paramaguru*, by which term he generally refers to him in his works:

यतो भार्गवपरमेश्वराचार्येण अस्मत्परमगुरुणा 'चलांशास्स्वं' (4546) इति कल्यब्दे परीक्ष्य पञ्चदशांशपूर्तिर्निर्णीता।[20];
अस्मत्परमगुरुणापि सिद्धान्तदीपिकायाम् एतत् प्रतिपादितम्।[21]

Date of Nilakantha

Indisputable evidences are available for fixing the date of our author. Sankara, Nilakantha's pupil, in his commentary on his teacher's *Tantrasangraha,* points out that the first and last verses of that work contain chronograms specifying the dates of the commencement and of the completion of the work. Thus, after giving the literal meaning of the first verse of the work[22]:

'हे विष्णो निहितं कृत्स्नं' जगत् त्वय्येव कारणे।
ज्योतिषां ज्योतिषे तस्मै नमो नारायणाय ते।।

Sankara says :

आचार्येण इमं श्लोकं आदितो ब्रुवता प्रथमपादेन प्रबन्धारम्भदिनकल्यहर्गणश्च अक्षरसंख्यया उपदिष्टः। समाप्तिसमयाहर्गणश्च 'लक्ष्मीशनिहितध्यान' इत्यन्ते भविष्यति।

These two *Kali* dates, 16,80,548 and 16,80,553, work out to *Kali*

[19] *Siddhanta-darpana-vyakhya*, on verse 27.

[20] *Ibid.,* under verse 18.

[21] *Aryabhatiyabhashya*, Golapada, verse 3.

[22] *Tantrasangraha*, verse 1, refer to *fn*.37.

[23] These correspond in the Gregorian calender to March 22, 1500 and March 27, 1500 AD respectively.

year 4601, *Mina* 26, and 4602, *Mesha* 1, both dates[23] occurring in AD 1500.

The *Siddhantadarpana* and Nilakantha's own commentary thereon give, respectively, the year and actual date of his birth[24],

कलिसन्ध्यष्टमांशे स्वशतांशाढ्ये गते ततः।
धनुर्मिथुनयोर्मध्ये प्रायशस्त्वयने उभे॥

दिव्याब्दशतमिता खलु काले सन्ध्या स्मर्यते। तस्य अष्टमांशः सार्धदिव्याब्दद्वादशकः। स च सौराब्दानां पञ्चचत्वारिंशत्-शतम् (4500)। तस्य शतांशः पञ्चचत्वारिंशदब्दः (45)। ततः स्वशतांशाढ्यः 'शिवशिवे' ति (4545)। कल्यब्दैस्तावति याते उभे अयने उत्तरदक्षिणाख्ये प्रायशो धनुर्मिथुनमध्ये स्तः। तदा अयनचलनांशाः धनात्मकाः पञ्चदशसंख्या बभूवुः। प्रायिकत्वं च कलाष्टकाधिकत्वात्। यतो भार्गवपरमेश्वराचार्येण अस्मत्परमगुरुणा 'चलांशास्स्वं' (4536) इति कल्यब्दे परीक्ष्य पञ्चदशांशपूर्तिर्निर्णीता। अतः सन्ध्याष्टमांशशतांशस्य प्रायिकत्वम्। स्वजन्मकालज्ञापनार्थं चैवमुक्तम्। तदा अहर्गणश्च 'त्यजाम्यज्ञतां तर्कैः' (16, 60, 181) इति। (p. 17 of the edn).

Here, Nilakantha himself says that he was born on the *Kali* day 16,60,181, which corresponds to June 17, 1444 AD, according to Gregorian calendar.

That Nilakantha lived to a ripe old age, even to become a centenarian, is attested by a contemporary reference made to him in a Malayalam work on astrology, *viz.*, the *Prasnasara* by Madhava, a Namputiri Brahman of the Incakkazhava house in Kerala, who wrote his work in AD 1542-43. Here, Madhava says that he could count upon reputed authorities like 'Kelanallur' to recommend his work.

Aalayat-tadaravil adiyil Attimattam
lokottaran punar-itinn-iha 'Kelanallur' |
Aabhasar allarivatullavar adarippan
porum prasiddhi perikollavar untanekam ||

The date of composition of this work, *Prasnasara,* is given as Kollam era 718/*Kali* 4644 (AD 1543) by the following verse in the work itself:

ezhunuttorupattettavatu kollam ataya na |
varunna visuvad, bhavatattvam (4644) *kalyabdam ayatu* ||

Rightly does Nilakantha remark in his *Aryabhatiyabhashya*[25]:

मयाद्य प्रवयसा . . . . यथाकथञ्चिदेव व्याख्यानमारब्धम्।

Moreover, we know of at least two more works composed by him

[24] *Siddhantadarpana*, verse 18, refer to *fn.*35.
[25] *Aryabhatiya,* TSS No. 101 (Trivandrum, 1930), p. 156.

subsequent to his writing the *Aryabhatiyabhashya, viz.*, the commentary on the *Siddhantadarpana* and the *Jyotirmimasa,* both of which quote the *Aryabhatiyabhashya.*

Versatility of Nilakantha

For a *Jyautishika,* and one who had specialised on its astronomical aspect, Nilakantha seems to be very well read. Every other page of his writings substantiate his knowledge of the several branches of Indian philosophy and culture. Sundararaja, the Tamil astronomer, calls him *Shad-darsana-parangata,* 'one who had mastered the six systems of philosophy'. Nilakantha himself informs us that he studied *Vedanta* under Ravi: cf., *Ravita atta-vedanta-sastrena.* He can refer to a *Mimamsa* authority to establish a mathematical point[26] and with equal felicity apply a grammatical dictum to the same purpose.[27] Pingala's *Chandas-sutra*[28] and the lexicons are quoted as the occasion demanded. The scriptures and the *Dharmasastras* also come in for citation.[29] And, so also the *Puranas*[30] like the *Bhagavata*[31] and the *Vishnu*.[32]

As for *Jyotisha* works, Nilakantha exhibits a surprising familiarity with a large number of them, from the *Vedanga-jyotisha* down to the treatises of his own times. He uses all types of *Jyotisha* texts, *Ganita, Samhita* and *Hora,* but as became his subject of specialisation, his quotations are mainly from texts dealing with astronomy proper. Some of the more important texts of all-India prevalence that Nilakantha quotes are: *Vedanga-jyotisha, Aryabhatiya,* Varahamihira's *Pancasiddhantika, Brihajjataka* and *Brihatsamhita, Suryasiddhanta,* Sripati's

[26] Cf., *Aryabhatiyabhashya,* TSS No. 101, pp. 54, 158, where Parthasarathi Misra's *Vyaptinirnaya* and *Advaitavivarana* and *Ajita* (commentry on *Slokavarttika*) and its commentary *Vijaya* come in for quotation. On *Golapada*, 50, the *Brihattika* of Kumarila Bhatta is cited.

[27] Cf., quotations from the *Vakyapadiya, Aryabhatiyabhashya,* TSS No. 110 (Trivandrum, 1931), p. 31.

[28] See *Aryabhatiyabhashya,* TSS No. 101 (Trivandrum, 1930), p.4.

[29] See commentary on *Siddhantadarpana*, verses, 1,2; the *Grahana* work, pp. 48, 49: and *Aryabhatiya-bhashya, Golapada,* verse 48, where the *Taittiriya-Aranyaka, Rigveda, Parasarasmriti, Kalanirnaya* of Sayana, *Manusmriti* etc. are quoted.

[30] See *Siddhantadarpana-vyakhya*, p.1, refer to *fn.*35.

[31] Cf., *Aryabhatiyabhashya,* TSS No. 110 (Trivandrum, 1931), pp. 16, 26.

[32] Cf., *ibid.*, p.8.

Siddhantasekhara and Manjula's *Laghumanasa*. Of the texts popular mainly in Kerala may be mentioned, the *Parahitaganita* or *Grahacara-nibandhana* of Haridatta, the *Bhashya* on the *Aryabhatiya* and *Laghu* and *Mahabhaskariyas* by Bhaskara I, Govindasvami's *Bhashya* on the latter and Paramesvara's super-commentary thereon. Other works of Paramesvara like his *Aryabhatiya-vyakhya* also come in for citation as also passages from his own teacher Damodara. Another Kerala author whom Nilakantha quotes often is Madhava, often styled *Golavid*, who was a reputed astronomer of the times.[33] Manuscripts of several works quoted by Nilakantha are yet to be unearthed and a detailed study of the numerous authorities quoted by Nilakantha is bound to throw welcome light on the annals of Hindu astronomy.

Nilakantha's works

Nilakantha has written several works which reflect his deep study of and ripe scholarship in astronomy, embodying the results of his investigations in the subject and interpreting the science lucidly.

i. Golasara[34]

The *Golasara* is a short work in 56 *arya* verses, divided into three *paricchedas*, containing, respectively, 11, 15 and 30 verses. The first *pariccheda* sets out the basic astronomical constants, *viz*., the number of civil days and the revolutions of the planets in an aeon, the positions of the higher apses and the ascending nodes of the planets, their maximum latitudes, their epicycles to the equations of the apses and of conjunction, the diameters of the orbits of the Sun and the Moon and the *yojana* measures of the epicycles.

Pariccheda II is concerned with the presentation of the celestial globe (*Jyotirgola*) from the point of view of astronomical conceptions and observations and the movement, therein, of the heavenly bodies. The position of the great circles, *Ghatika-mandala* (celestial equator) and the *Apakrama-mandala* (ecliptic), their mutual obliquity, the division of the

[33] On this Madhava (c.1340-1425), who was a teacher of Paramesvara, see the present writer's introduction to Madhava's *Venvaroha* (Trippunithura, Cochin, 1957), and *Sphutacandrapti* (Hoshiarpur, 1973).

[34] Critically ed. with Introduction by K.V. Sarma, Vishveshvaranand Institute, Panjab University, Hoshiarpur, 1970.

ecliptic, the (apparent) rotation of the celestial globe, the rising point of the ecliptic (*lagna*), the measure of the orbits of the planets, the measurement of the positions of the planets on the ecliptic and the position of the horizon at the equator and elsewhere are noticed here, in order.

In *Pariccheda* III, verses 1 to 15 deal with the circle and the graphical and computational derivation of the sines. Verses 16 to 30 discuss the inter-relationship of the *Manda, Sighra* and *Kakshya* circles of the different planets and also how the results arrived at by calculation of the positions of the planets are affected by their *Kshepa* (deflection) from the ecliptic.

ii-iii. Siddhantadarpana and Vyakhya[35]

The importance of *Siddhantadarpana* lies in the fact that the author presents herein the astronomical constants as verified through his own observations and investigations. The passages expressing these constants having been adopted and commented upon by the author in his commentary, which latter happens to be one of his last works. It could be taken that the values of the constants as given herein are the final figures accepted by Nilakantha.

In the first part of the work called *Upadesa-bhaga* (Theory section), the author enunciates in twenty couplets (2-21), his view on the number of revolutions of the planets, their higher apses (*mandocchas*) and ascending nodes (*patas)* during a definite period of time, the dimensions of the epicycles of equations of the apses (*manda-paridhi*) and of conjunctions (*sighra-paridhi),* the measure of the aeons (*yugas*), the velocities of the planets, the measures of the diameters of the Moon and the Sun, the geographical location of the city of Avanti, the situation of the ecliptic (*apakrama-vritta*), and the conception of the epicycles.

In the second part called *Nyaya-bhaga* (Practical section) (verses 21-31), are set forth the eccentric and orbital circles *(pratimandala* and *kakshya-mandala),* the sines etc. of the angles measured on these circles, the geocentric positions of the planets, declination and its measurement. The author also gives his views on the occurrence of the *vyatipatas,* the

[35] Critically ed. with the author's own commentary and Translation and two Appendices and detailed Introduction by K.V. Sarma, Vishveshvaranand Institute, Panjab University, Hoshiarpur, 1976.

extent of the lunar crescent and the data from which eclipses are to be calculated.

The commentary

In line with his *Bhashya* on the *Aryabhatiya,* Nilakantha's commentary on the *Siddhantadarpana* is elaborate and discursive. Alongside explaining the text proper, it introduces related topics by way of background, illustration and rationalisation. Often, the commentary dilates into verse, a common practice with Kerala astronomers like Paramesvara. This feature is found also in Nilakantha's *Aryabhatiyabhashya.*

Nilakantha's discussions are often highly instructive. An instance in point is his detailed analysis of the mental working of a mathematician who proceeds to derive the relation between the sides and the hypotenuse of a right-angled triangle (pp. 22-24). In the course of this discussion, he makes a mention also of the two methods of approach for solving mathematical problems, *viz.,* that of logical reasoning and that of demonstration on the board. He adds that one should first try the method of demonstration, for logical reasoning is limited, endless and sometimes inconclusive (*alpa-vishayatvat, anantyat, kvacidapyavisrante'sca,* p.23). Equally instructive are the methods of constructing the armillary sphere, the defining of the situation of *rasis* therein (p. 14), and the exposition of the computation of the geocentric positions of the planets (pp. 25-31).

The following observations of Nilakantha are noteworthy :

(i) *Ayanamsa* (measure in minutes of the precession of the equinoxes) should be derived by observation, if it has to be accurate (p. 17).

(ii) The use of *Trairasika* (Rule of three) is justified even in computations concerning moving bodies, since the factors involved are considered only for specific moments at which they might be taken as stationary (p. 25).

(iii) In astronomical computations, it does not make any difference whether the eastward motion of the Earth or the westward motion of the planets are taken, because the motion is relative (p. 5).

(iv) The *Bhagola* (*Rasigola*, ecliptic sphere) revolves as a whole and not *Vayugola* (having the *Ghatikamandala* or Celestial equator as one of its great circles); thus the relative distances between the stars remain constant, from which other results follow (pp.16-17).

iv,v. Candracchayaganita and Vyakhya[36]

Chayaganita or 'Computations concerning the Shadow' constitutes the Hindu astronomer's method for ascertaining the exact time of occurrence of any event on the basis of the shadow cast by the Sun or the Moon. The expediency of this device during ancient and mediaeval times would be apparent when one considers the limitations of such chronometrical devices as the hour-glass, water-disc, etc. The gnomon, being one of the simplest of astronomical instruments that could be set up at any time and at any place, and readings made independent of circumscribing factors like zero-point etc., is the most handy instrument to record such abrupt occurrences as the birth of a child, death of a person, etc. A knowledge of the exact moment of these happenings, as computed by astronomical methods from the measure of the shadow, facilitates the prediction of the future based on them in accordance with the dictums of astrology. Obviously, the shadow cast by the Sun is measured during day and that by the Moon during night.

The *Candracchayaganita* of Nilakantha Somayaji, belongs to this genre of texts and sets out the processes for the computation both of *Kramacchaya,* 'Shadow from Time' (verses 1-17), and *Viparitacchaya,* 'Time from Shadow' (verses 18-32), of the Moon. The commentary by Nilakantha himself gives a lucid exposition of the textual verses.

Kramacchaya and Viparitacchaya

The utility of *Chayaganita* in the daily life of the mediaeval Hindu, whose prescribed way of life required the perfomance of numerous religious rites, is not far to seek. Ordinary astronomical computation enabled him to calculate auspicious times *(muhurta)* for sacred rites and social functions, in terms of time-units like *nadikas* and *vinadikas*, say, after Sunrise or Sunset. But, in the absence of accurate chronometers, it was not easy to ascertain when that auspicious moment, as calculated, had arrived. *Chayaganita* came to his rescue in such a situation. For, it was possible to calculate, in advance, the length that the Sun's or the Moon's shadow would attain at the appointed time. One could, then, set up the gnomon and watch for the shadow to reach the stipulated length and perform the rite at the right moment. This process of computing the shadow

[36] Cr. Ed. with Auto-commentary, Introduction and two Appendices by K.V. Sarma, Vishveshvaranand Institute, Hoshiarpur, 1976.

for any specific time is called *Kramacchaya*, 'Direct (process of) Shadow (computation)'.

The converse of *Kramacchaya*, is known as *Viparitacchaya*, 'Reverse (process of) Shadow (computation)', according to which time is calculated from the length of the shadow. This was equally, if not more, important in everyday life, in that it enabled the accurate ascertainment of the time of abrupt, non-planned occurrences like moments of birth and death, unexpected arrivals, untoward happenings and the like.

Popularity of Chayaganita in Kerala

The mediaeval astronomers of Kerala seem to have taken the best advantage of this natural phenomenon and devised highly intricate calculations to get the exact times corresponding to the measures of the shadow, taking into consideration also the factors that affected the shadow, such as the latitude and longitude of the place, time of the year, precession of the equinoxes, etc. Practically every one of the numerous astronomical manuals *(Karana-granthas)* produced in Kerala contains sections devoted to *Chayaganita*, both of the Sun and of the Moon. In the *Pancabodha* manuals, of which about a dozen different texts have been identified and documented, *Chaya* is one of the five subjects dealt with, the others being *Vyatipata, Grahana, Sringonnati* and *Maudhya*. Several texts which are devoted solely to computations based on shadow have also been composed.

vi. Tantrasangraha[37]

Tantrasangraha, composed on the model of the '*Tantra*' class of astronomical texts in which the zero point for computation is taken as the commencement of the *Kaliyuga* as in texts likes the *Aryabhatiya*, is a major work of Nilakantha. Perhaps, to be in tune with tradition, Nilakantha makes use of the *bhutasankhya* notation of all-India prevalence in place of the Kerala system of *katapayadi*, which he uses in his other works. In about 432 verses, mostly couched in the *anushtubh* metre, the author depicts the entire gamut of theoretical and practical astronomy in eight sections entitled *prakaranas*: I. *Madhyama* (Mean planets), II. *Sphuta*

[37] Cr. Edited with two commentaries, *Yuktidipika* and *Laghuvivritti*, and detailed Introduction and Indices by K.V. Sarma, Vishveshvaranand Institute, Panjab University, Hoshiarpur, 1977.

(True planets), III. *Chaya* (Gnomonic shadow), IV. *Candragrahana* (Lunar eclipse), V. *Ravigrahana* (Solar eclipse), VI. *Vyatipata* (The 'aspect' of the Sun and the Moon when the sum of their longitudes is 180^0 and their declinations (*kranti*) are equal in magnitude and direction), VII. *Drikkarma* (Reduction to observation), and VIII. *Sringonnati* (Elevation of the Moon's horns).

The initial verse of the work gives, in a chronogram, the date of the commencement of the work and the last verse, in another chronogram, the date of completion of the work, both in 1500 AD, with five days intervening between them; while this may seem to be too short for the composition of the work, it need not be so; for a literary giant and idea-filled astronomer this is not a difficult feat.

Through *Tantrasangraha,* Nilakantha explores new fields, formulates novel ideas and introduces major revisions as in the case of the inner planets Mercury and Venus in the matter of their planetary model and the better formulation of their equation of centre.

The two commentaries, *Yuktidipika* and *Laghuvivritti* both by Sankara, on the *Tantrasangraha* are highly useful in understanding the text, especially the former. It is, indeed, a real *yuktidipika* in that it offers lucid rationales to the statements in the text.

Manuscripts of Tantrasangraha

The *Tantrasangraha* is divided into eight chapters, and contains 432 verses in all. Since the work had been popular and was accepted as an authority, a large number of manuscripts of the work exist, of which about fifty have been documented in the New Catalogus Catalogorum, (Madras University, Vol.VIII, 1974, p.97). One of these manuscripts, No. 475-E of the Kerala University Oriental Research Institute and Manuscript Library, gives the date of its transcription through the chronogram *sevyo dugdhabdhitalpah,* being *Kali* day 16,99,847, falling in the year AD 1551, and so this Ms. is almost contemporaneous with its author Nilakantha Somayaji who passed away in about AD 1545. All the said documented manuscripts conform to the division and extent of the work as mentioned above. In several cases there are post-colophonic statements expressing the completeness of the work in eight chapters. In one such manuscript, dated M.E. 928 (AD 1753), being Ms. No. C. 224-C, deposited in the said Library, the total number of verses in the work is also stated to be 432, which conforms to the presently available text.

इति तन्त्रसंग्रहे अष्टमोऽध्यायः। तन्त्रसंग्रहः समाप्तः। अध्यायं एट्टिलुं कूटि लोकङ्ङळ् नानूट्टिमुप्पत्तिरण्टु। 928-ामाण्ट चिङ्ङमासत्तिल् एषुत्तिच्च पुस्तकम्॥

vii. Aryabhatiyabhashya[38]

Aryabhatiyabhashya is an elaborate commentary on the cryptic and *sutra*-like text of *Aryabhatiya* which comprehends in 121 *aryas* the fields of Mathematics and Astronomy. A perusal of the commentary will amply prove that it is no false claim that Nilakantha makes when he designates his work as a `*Mahabhashya*' and explains the method of exposition adopted by him[39]:

श्रीमदार्यभटाचार्यविरचितसिद्धान्तव्याख्याने 'महाभाष्ये' उत्तरभागे युक्तिप्रतिपादनपरे त्यक्तान्यथाप्रतिपत्तौ निरस्तदुर्व्याख्याप्रपञ्चे समुद्घाटितगूढार्थे सकलजनपदजातमनुजहिते निदर्शितगीतिपादार्थे सर्वज्योतिषामयनरहस्यार्थनिदर्शके समुदाहृतमाधवादिगणितज्ञाचार्यकृत-युक्तिसमुदाये निरस्ताखिलविप्रतिपत्तिप्रपञ्चसमुपजनितसर्वज्योतिषामयनविदमलहृदय-सरसिजविकासे निर्मले गम्भीरे अन्यूनातिरिक्ते गणितपादगतार्यात्रयस्त्रिंशद्व्याख्यानं समाप्तम्।

In another context, recalling how he came to write the commentary, Nilakantha remarks[40]:

मयाद्य प्रवयसा ज्ञाता युक्तीः प्रतिपादयितुं भास्करादिभिरन्यथाव्याख्यातानां कर्माण्यपि प्रतिपादयितुं यथाकथञ्चिदेव व्याख्यानमारब्धम्।

The lucid manner in which the difficult conceptions about the celestial globe and astronomical calculations are made clear, the wealth of quotations, and the results of personal investigations and comparative studies presented herein amply justify the appelation *Mahabhashya* which Nilakantha has given to his work.

Nilakantha has commented only on the *Ganita*, *Kalakriya* and *Gola-padas* of the *Aryabhatiya*, leaving out the *Gitikapada*, which he says is covered by the commentary on the other three sections[41]:

तत्रेयं त्रिपाद्यस्माभिर्व्याचिख्यासिता, यतस्तद्व्याख्येयरूपत्वाद् गीतिकापादस्य। एतद्व्याख्यानेनैवार्थः प्रकाशेत।

[38] Ed. in 3 volumes, TSS Nos. 101, 110, 185 (Trivandrum, 1930, 1931, 1957).

[39] See *Aryabhatiyabhashya,* TSS No. 101 (Trivandrum, 1930), p.180.

[40] *Ibid.*, p.156.

[41] *Ibid.*, p.1.

viii. Grahananirnaya

This is a work on the computation of lunar and solar eclipses. Manuscripts of this work are yet to be discovered, but later authors and Nilakantha himself in his *Aryabhatiyabhashya* quote from this work[42]:

तदैव ग्रहणमध्यं च। स्फुटसाम्ये तु विक्षेपकोटिमण्डलापक्रममण्डलयोः भुक्तभागसाम्यमेव स्यात्। तदुक्तं मया ग्रहणनिर्णये-

परमक्षेपकोटिघ्नः पातोनार्कभुजागुणः।
स्वेष्टविक्षेपकोट्याप्तस्तत्क्षेपकृतियोगतः।।
पदं यच्चापितं यच्च पातोनार्कभुजाधनुः।
तद्विशेषं हतं षष्ट्या गत्यन्तरहृतं क्षिपेत्।।
पर्वान्ते युक्पदे क्षेपे शोधयेद् विषमे पदे।
एवं कृतोऽपि पर्वान्तः सूर्येन्द्वोर्ग्रहणे स्फुटम्।।

These verses are quoted also by Sankara in his commentary *Laghuvivritti* on Nilakantha's *Tantrasangraha*[43] with the introductory remark: तदुक्तमनेनैव ग्रहणनिर्णये।

ix. Sundararaja-prasnottara

Sundararaja, son of Anantanarayana, was an astronomer of the Tamil country contemporaneous with Nilakantha and author of a detailed commentary on the *Vakyakarana* or *Vakyapancadhyayi* which is a manual on the basis of which almanacs are computed in the Tamil districts.[44] Sundararaja had the greatest respect for Nilakantha whom he addressed for clarification of certain points in astronomy. Nilakantha's detailed answers to these questions formed a regular work, *Sundararaja-prasnottara.* Manuscripts of this work are yet to come to light, but both the authors refer to this work. Sundararaja in his commentary on the last verse of Ch.V of the *Vakyakarana* says:[45]

अत्र तु गतियोगांशकेनैव हरणं युक्तमिति श्रीमत्-केरलसद्ग्रामनिवासि-नीलकण्ठार्येण त्रिस्कन्धविद्यापारदृश्वना षड्दर्शनीपारंगतेन आश्वलायनसूत्रेण गार्ग्यगोत्रेण नवकलरु(?) जातेन गोलचूडामणिना अस्मदनुग्रहार्थे सुन्दरराजप्रश्नोत्तराख्ये ग्रन्थे प्रतिपादितम्। तेन गतियोगेनैव विभज्य स्थितिदलं ज्ञेयम्।

[42] See *Aryabhatiyabhashya,* TSS No. 185 (Trivandrum, 1957), p.102.

[43] In Chapter IV, verse 27, TSS No. 188, p.107.

[44] Cr. ed. with Introduction and Appendices by T.S. Kuppanna Sastri and K.V.Sarma, K.S.Research Inst., Madras, 1964.

[45] *Ibid.,* p.119.

Nilakantha too refers to this work in his *Aryabhatiyabhashya*[46],

सुन्दरराजप्रश्नोत्तराख्ये मयोक्तमत्राप्यनुसन्धेयम्।

x. Grahaparikshakrama

The well-known Kerala astrologer, the late Puliyur Purushottaman Namputiri, has edited[47] an old incomplete Malayalam summary of a Sanskrit work under the title *Grahaparikshakrama.* The textual verses were not available to the editor and he presumed that the author was Drigganita-Paramesvara. These verses are, actually, to be found in Nilakantha's *Bhashya* on the *Golapada* of the *Aryabhatiya*[48]. It is a long tract with about 200 verses, enunciating the principles and methods for verifying astronomical computation by regular observation. The work, as available, ends thus:

इति संक्षेपत: प्रोक्ता परीक्षा ज्योतिषामिह।
कालमानचतुष्कस्य श्रुतस्य विवृतिस्त्वियम्।।

xi. Jyotirmimamsa[49]

The *Jyotirmimamsa* of Nilakantha is a unique work, the like of which does not seem to exist in Indian astronomical literature. It asserts that astronomy is primarily a discipline of observation, experimentation and computation and deserves to be revised periodically if the results are to be true. The processes to be employed and comparative studies to be made in different cases have also been duly indicated. For this reason, the work deserves to be set out in greater detail than the other writings of Nilakantha.

a. *Real implication of 'divine instruction'*

At the outset, Nilakantha draws attention to two eclipses which actually occurred during his times, but whose presence would not have been detected had they been computed through the astronomical constants given

[46] See *Aryabhatiyabhashya,* TSS No. 185 (Trivandrum, 1957), p.149.

[47] Published by the Astrological Research Institute, Bombay-25, 1950.

[48] See *Aryabhatiyabhashya,* TSS No. 185 (Trivandrum, 1957), pp.132-49.

[49] Ed. with Introduction and Appendices, by K.V.Sarma. Vishveshvaranand Institute, Panjab Uniersity, Hoshiarpur, 1977.

in the *Gitikapada* of the *Aryabhatiya.* Nilakantha explains that it was for this very reason that Aryabhata intended astronomy to be a practical discipline and recommended the verification and revision of his own astronomical constants by observation, towards which he had also suggested a method in *Golapada,* 48 (pp.1-2). To a possible query that the number of planetary revolutions given by Aryabhata are immutable since they form part of 'divine instruction', Nilakantha points out, pertinently, that by the expression 'divine instruction' is not meant any direct instruction by the gods, but only the chastening of the intellect through divine grace, as a result of which the author could express his thoughts logically (pp.2-3).

b. *Jyotisha, an observational science*

Nilakantha then takes pains to demonstrate the place of observation and logical inference in the maintenance and furtherance of astronomical tradition (pp. 6-10). A passage from a *Mimamsa* text, which Nilakantha quotes in this context, fully expresses his idea:

> The correlation, of the computed Moon etc. with actual observation at a particular place, the revision of computation on the basis of such correlation, logical inference therefrom being transmitted as tradition, it being again correlated (with observation and again revised) and transmitted further down to others – this is how tradition is continued without interruption, and hence its (continued) authoritativeness[50].

In the same vein, Nilakantha argues:

> One has to accept that (each of) the five *Siddhantas* had been authoritative at one time (or other, though they might not be so, now). Therefore, one has to look for a system which tallies with observation. The said tallying has to be verified by contemporary experimenters at the time of eclipses. When the two vary, i,e., when an earlier system does not agree (with observation), experiments have to be conducted with instruments, and revolution numbers of the planets calculated therefrom. A new system has thus to be expounded. Nobody will be ridiculed for this in this world nor punished in the next[51].

[50] गणितोन्नीतस्य चन्द्रादे: देशविशेषान्वयस्य प्रत्यक्षेण संवाद:, ततो निश्चितान्वयस्य परस्य गणितलिंगोपदेश:, ततस्तस्याप्तोपदेशावगतान्वयस्य अनुमानं, संवाद:, परस्मै चोपदेश: इति सम्प्रदायाविच्छेदात् प्रामाण्यम्।

[51] पञ्चसिद्धान्तास्तावत् क्वचित्काले प्रमाणमेव इत्यवगन्तव्यम्। अपि च य: सिद्धान्तो

c. *Corrections to planetary parameters*

Having established the necessity of revising planetary parameters, periodically, Nilakantha presents the correcticns propounded to the parameters of *Aryabhatiya* at different times by certain astronomers including Haridatta, Govinda, Brahmagupta and Lalla (pp. 10-12).

A point of interest centres round two sets of corrections, one by Haridatta, commencing from *Saka* 444 (*Kali* 3623) as given in his *Grahacaranibandhana,* and the other by Lalla, commencing from *Saka* 420 (*Kali* 3599) as given his *Sishyadhivriddhida.* Both these corrections are presumed to have commenced from the date of composition of the *Aryabhatiya* given in verse 10 of the *Kalakriyapada.* Nilakantha points out that the discrepancy between the dates as accepted by the two authors has arisen on account of the different interpretations, the former taking *Kali* 3600 mentioned in the verse as the date of birth of Aryabhata and the latter as the date of composition of the *Aryabhatiya.* While discussing the correction advocated by Haridatta, Nilakantha notes that his interpretation was wrong and that *Kali* 3600 was actually the date of composition of *Aryabhatiya* (pp. 13-14).

Nilakantha takes the occasion to draw attention also to the apparent error in Lalla's commencing the correction from *Saka* 420 (*Kali* 3599) instead of from *Saka* 421 (*Kali* 3600), but justifies it on two grounds: (1) 420 which is a round figure is more amenable to mathematical operations and increases the resultant but by one, whatever be the year for which readings are taken, causing thereby but negligible difference in the result. (2) In the case of the Moon such an addition is actually warranted. Especially for the second reason, Nilakantha goes to the extent of saying that Lalla had used the figure 420 intentionally for this purpose (pp.14-17).

A similar justification has been made also by Suryadeva-yajvan (born AD 1191) in his commentary on the *Aryabhatiya.* Nilakantha extracts the entire section from Suryadeva in order to buttress his view (pp. 19-20).

दर्शनाविसंवादी भवति सोऽन्वेषणीयः। दर्शनसंवादश्च तदानीन्तनैः परीक्षकैः ग्रहणादौ विज्ञातव्यः। ये पुनरन्यथा, प्राक्तनसिद्धान्तस्य भेदे सति यन्त्रैः परीक्ष्य ग्रहाणां भगणादिसंख्यां ज्ञात्वा अभिनवसिद्धान्तः प्रणेय इत्यर्थात्, तत् त इहलोकेऽहसनीयाः, परलोकेऽदण्डनीयाश्च इति।

d. *Reason for different Means for same revolutions*

The author poses a question as to how the means of certain planets calculated according to different systems differ even though the numbers of their revolutions per *Kalpa* as given in the said systems are the same. In answer, it is demonstrated that it is so because the number of civil days in *Kalpa* differs in the different systems, resulting in a corresponding difference in the means also (pp.17-19).

e. *Vedic authority for inference in astronomy*

Earlier in the work (pp. 3-6), Nilakantha had stressed the significance of logical inference in the matter of astronomical computation. Vedic approbation in the matter is sought to be provided by a reference to a passage from the *Taittiriya-aranyaka,* 1.2.1, as expounded by the well-known Vedic scholiast Sayana-Madhava in his work *Kalanirnaya.* It is specified in the said passage that *Smriti, Pratyaksha, Aitihya* and *Anumana* (Inference) are means to determine the periods of planetary motion (pp. 21-22).

f. *Relative accuracy of different systems*

For an estimation of the relative accuracy of the different systems, Nilakantha adopts the ingenious method of computing the planets for a common date. For this purpose *Kali* day 16,82,112 is chosen, it being a date occuring in AD 1504 and, so, contemporary to his own times. This date has the advantage also of being 8/75,00,000 of a *Kalpa*, which makes calculations easy. The results obtained are then correlated to observation.

In this connection, Nilakantha takes into consideration the Sunrise and Midnight systems (*Audayika* and *Ardharatrika pakshas*), both of Aryabhata, and those of Brahmagupta and Sripati, as set out in *Brahmasphutasiddhanta* and *Siddhantasekhara,* composed respectively by the latter two authors (pp. 22-31).

g. *Correction through eclipses*

Well defined eclipses being the most convenient natural phenomena for the verification of astronomical systems and effecting corrections thereto, the utilisation thereof is stressed by Nilakantha. He instructs as follows, in this connection:

> The eclipses illustrated in the *Siddhantadipika* might be computed (in all their details) by the investigator. Otherwise, other eclipses handed down by the tradition of one's own school might be computed. In the light of these and of the eclipses actually observed by oneself, future eclipses should be computed and forecast. Or, eclipses occurring at other parts of the country should be computed using the longitude, latitude, etc., of that part of the country and, with that as basis, the true Sun, Moon and Node (at the relevant times) ascertained. Then, from the Sun, Moon and Node, ascertained (as above), past and future eclipses at one's own place shall be computed, using the latitude and longitude of one's own place[52].

Nilakantha follows up the above instruction with the details of a number of eclipses observed and recorded by Parameswara, his grand-teacher, in the *Siddhantadipika*[53] (pp. 32-35).

At the close of the enumeration of the eclipses, Nilakantha indicates the procedure for verification. He says :

> The computation of the Mean planets having been enunciated by Paramesvara later than Sripati, the former will tally (better) with observation. Therefore, in the case of the eclipses enumerated by Paramesvara, those observed by me (Nilakantha) and others that might be mentioned hereafter, the Mean Sun etc., shall be computed as directed by Paramesvara. Their True positions shall (however) be computed according to Sripati's method. The eclipses should then be computed duly making use of the methods derived from the rationales enunciated by me (in my *Bhashya*) on the *Kalakriya* and *Gola padas* (of the *Aryabhatiya*) [54].

Nilakantha then illustrates his instructions by means of a practical example (pp.35-36). The precession of the equinoxes which has also a bearing on the results is also touched upon here (pp.36-37).

[52] अथ सिद्धान्तदीपिकायाम् उदाहृतानि ग्रहणानि परीक्षमाणेन गण्यानि। अन्यानि वा ऐतिह्यसिद्धानि स्ववंश्यैरुपदिष्टानि गण्यन्ताम्। एतैः प्रत्यक्षावगतैश्च अनागतग्रहणं निर्णीय वक्तव्यम्। जनपदान्तरजात् तद्देशान्तराक्षादिभिः गणयित्वा रवीन्दुतुंगपातान् विज्ञाय अतीतमनागतं स्वदेशजं ग्रहणं स्वदेशान्तराक्षाभ्यां विज्ञातैस्तैरेवं ग्रहोच्चपातैर्गणयित्वा निर्णेयाः।

[53] See *Siddhantadipika*, the super-commentary of Paramesvara on Govindasvami's *Bhashya* on the *Mahabhaskariya* of Bhaskara, under verse 5.77 (Ed. by T.S.K. Sastri, Madras, 1957, pp. 329-32).

[54] सिद्धान्तशेखराद्युक्तमध्यमेभ्यः परमेश्वरोक्तानां नूतनत्वाद् अस्य दृष्टिसाम्यं स्यात्। तस्मात् सिद्धान्तदीपिकोदाहृतानि अस्माभिः (नीलकण्ठेन) दृष्टानि च तत्तदवसरे वक्ष्यमाणानि परमेश्वरोक्तप्रकारेण अर्कादिमध्यमान्यानीय श्रीपत्युक्तप्रकारेण स्फुटीकृत्य कालक्रिया-गोलपादोक्ताभिः अस्माभिर्व्याख्याताभिः युक्तिभिः सिद्धैः क्रियाविशेषैश्च गण्यन्ताम्।

h. *True Motion, Position, etc., of planets*

While continuing the previous computation, the rationale underlying the determination of the true motion of a planet at a particular place at a particular moment is explained in detail. Examples are also worked out for true Sun and Moon to illustrate the rationale of the *dvitiya-sphuta* of planets (pp.37-41). Another topic treated in a similar manner relates to the correction of the periphery of the *manda* epicycle by means of the *natajya* (Rsine zenith distance) (pp.42-44). Other corrections to be applied to get the true planet, such as *ayanacalana* (precession of the equinoxes), *pranakalantara* and *caradala* (half ascensional difference), are also set out in turn with the tables of the Rsines of the latter two and the method of application thereof (pp.44-49).

i. *Relation of the sides and hypotenuse*

After explaning the rationale involved in determining the height of a lamp-post by means of two gnomons (pp.49-50), Nilakantha rationalises, from fundamentals, the graphical proofs for the relation between the sides and hypotenuse of a right angled triangle – the Pythagoras theorem. The discussion is, however, incomplete, the manuscript having a long break here (p.50).

j. *Reduction of angular distances*

Errors in results arise on account of the observer being stationed on the Earth's surface, while the basic measures relate to the centre of the Earth-sphere or to the horizontal plane thereof. These are corrected by reducing the basic angular distances to the *driggola* ('visible celestial sphere'). Nilakantha rationalises this type of correction with reference to the orb of the Moon (pp.51-52).

Conversely, in the case of the gnomonic shadow, it is the measure on the *driggola* that needs to be reduced to that of the *bhagola* ('sphere of the zodiac'). The rationalisation of the processes involved forms the last section in the present manuscript, which breaks off half-way during the middle of the said discussion (pp.53-55)

The only available manuscript of *Jyotirmimamsa* is incomplete, its beginning and end being lost, and some gaps occur in the middle. However, the nature of the work and the manner in which its author Nilakantha is presenting it could give one some idea of the portions lost.

k. *The beginning of the work*

In the original palmleaf manuscript,[55] the *Jyotirmimamsa* is inscribed in continuation of another work of Nilakantha, being the *Siddhantadarpana* with auto-commentary. In this manuscript, the final portion of *Siddhantadarpana*[56] along with the initial portion of *Jyotirmimamsa* is lost, the loss in the latter work being, presumably, not more than one folio.

A cue for the missing portion could be had from the tenor of the first section of the work (see below, pp.1-2), where it is established that appropriate corrections based on observation should be applied periodically to astronomical parameters etc., so that they might give results which would tally with observation. The available manuscript commences with arguments based on eclipses supporting the said thesis, introduced with the words *atha grahanam.* It may be presumed reasonably that the author must have, earlier to this, presented similar arguments to the same purpose, based on other visible astronomical phenomena like the setting of the planets, the heliacal rising of the signs, gnomonic shadow etc., with the introductory words, *atha maudhyam, atha lagnam, athacchaya* etc.

Besides the above, the lost portion should have contained one or more introductory verses through which the author would have uttered the usual invocation to his favourite deity and stated his purpose in composing the work.

l. *Missing portions in the middle*

Apart from the minor gaps, with which the manuscript abounds and which have mostly been filled tentatively, there occur in the manuscript at least three major omissions which could, obviously, not be filled and, which therefore, mar the continuity of reading. The first of these occurs towards the end (?) of section 14, on 'Astronomical corrections through eclipses', where an illustrative eclipse was being computed (p.37). The second omission occurs towards the beginning of section 19, on the 'Height of a

[55] Ms. No. p.975 of the Trivandrum Palace Collection, now deposited in the Kerala Univ. Or. Res. Inst. and Mss. Library.

[56] This lost portion contains the last five verses, 28 to 32, of the work and the commentary thereon, vide, edn. of *Siddhantadarpana* with auto-commentary, by K.V.Sarma, Hoshiarpur, 1976, p.31.

lamp-post by means of two gnomons', the close of the previous section also having been lost (p.49). A portion towards the close of section 20 and beginning of section 21 is also lost.

m. *Omission at the end*

The extant manuscript breaks off abruptly towards the middle of section 22, being a discussion on the reduction of the minutes of arc of the visible celestial sphere (*driggola*) to those of the zodical sphere (*bhagola*). There is no possibility of knowing, with exactitude, what more Nilakantha had intended to include in the work. The comprehensive title *Jyotirmimamsa* given to the work and of Nilakantha's apparent intention (1) to demonstrate that the discipline of *Jyotisha* is based on observation, and (2) to study the comparative accuracy of the different systems, would induce one to presume that the work contained all that is conveyed by the title and covered by the said intentions. Thus, after completing the topic at hand, *viz.*, *Grahasphuta* (True positions of planets), Nilakantha should have taken up for investigation other subjects like *Lagna* (Rising point of the ecliptic), *Chaya* and *Viparitacchaya* (Shadow and Inverse shadow), and *Candrasringonnati* (Elevation of the Moon's horns).

The School of Nilakantha

An important work based on *Tantrasangraha* is the *Yuktibhasha*, 'Rationale in the Malayalam language', whose sole aim is to rationalise the theories involved in the constants and computations occuring in the *Tantrasangraha*. Thus, after the benedictory verses, the work commences with the statement:

> *"avite nate tantrasangrahatte anusariccu grahagatiyinkal upayogamulla ganitannale muzhuvanayi colluvan tutannunnetattu..."*

"Here, commencing an elucidation in full of the rationales of planetary computations according to the *Tantrasangraha....*"

The work finds its first reference in modern writings in an article by C.M. Whish in 1835, where it is referred to for verifying the date of the author of *Tantrasangraha*. Whish had contemplated "a farther (*sic*) account of the *Yuktibhasha*... will be given in a separate paper," which, however, does not appear to have been written or published.

Yuktibhasha, known also as *Ganitanyayasangraha,* 'Compendium

of astronomical rationale', has been a popular text in Kerala for more than four hundred years, since its composition towards AD 1530. Several manuscripts of the work are known. However, since the work is couched in Malayalam language which is spoken only in Kerala it has remained, practically, beyond the purview of scholars who did not know the language, in spite of its having been published. And the few articles on this important work relate to only certain individual topics treated herein. It is therefore necessary that a critical appraisal of the nature and contents of this work as a whole is made, so as to enable scholars take up the same for further study.

The introductory verses of the *Yuktibhasha* do not mention the name of its author, nor do its manuscripts indicate his name at their closing colophons. There are, however, evidences which point to the correct name of the author of *Yuktibhasha* as Jyesthadeva and his date to be AD 1500 – 1610.

Thus, an astronomical chronology (*granthavari)* in the Malayalam language found as a post-colophonic statement in an old palmleaf manuscript of a Malayalam commentary on *Suryasiddhantha* preserved in the Oriental Institute, Baroda, Ms. No. 9886, contains in it the statements:

> Paramesvara was a Namputiri from Vatasseri (family). He resided on the northern bank of the Nila (river)... His son was Damodara. Nilakantha Somayaji was his pupil. He, (the latter), is the author of the *Tantrasangraha, Aryabhatiyabhashya* and other works. His date is determined by the *Kali* days 16,80,553 (AD 1500).
>
> Jyesthadeva was the pupil of the above Damodara. He was a Namputiri from Parannottu (family). He is the author of the work *Yuktibhasha.* Acyuta Pisarati of Trikkantiyur was the pupil of Jyesthadeva. He is the author of the *Sphutanirnaya, Goladipika* and other works.
>
> Melaputtur Narayana Bhattatiri was the pupil of Acyuta Pisarati. He is the author of the *Narayaniya, (Prakriya) Sarvasva* and other works. His date is determined by the *Kali* days 17,12,210 (AD 1587).

In view of the facts that Damodara (c. 1400 – 1500) was a teacher both of Nilakantha Somayaji and Jyesthadeva and that Jyesthadeva wrote his *Yuktibhasha* in the wake of Nilakantha's *Tantrasangraha*, he must be a younger contemporary of Nilakantha. He is remembered in 1592 by his pupil Acyuta Pisarati as *Pravayas* ('very old'). His *Drikkarana* is dated 1608. Jyesthadeva should, therefore, have been long-lived, his date being

c. 1500 – 1610. His family house Parannottu (Skt. Parakroda) still exists in the vicinity of Alattur and Trikkantiyur where well known astronomers like Paramesvara, Nilakantha and Acyuta Pisarati flourished about those times.

Scope and Extent of Yuktibhasha

The entire text of *Yuktibhasha* occurs as one continuum, without any internal or closing colophons to mark off the subjects treated in the work. However, towards the middle of the work, where the treatment of mathematics ends and that of astronomy commences, occurs a general benedictory statement which reads: *Srirastu, harih, sri-ganapataye namah, avighnam astu.* This would naturally mean that the author had conceived his work as consisting of two parts, devoted respectively to mathematics and astronomy. Since the work deals with several main subjects and a number of topics under each, the needed subject and topic divisions shall have to be made editorially with suitable indication. Demarcating the work thus, the main subjects treated in Part I, Mathematics, are: *Parikarma* (Logistics), *Dasaprasna* (Ten problems involving logistics), *Bhinnaganita* (Fractions), *Trairasika* (Rule of three), *Kuttakara* (Pulverisation), *Paridhi-vyasa* (Relation between circumference and diameter) and *Jyanayana* (Derivation of Rsines). The subjects treated in Part II, Astronomy, are: *Grahagati* (Planetary motion), *Bhagola* (Sphere of the zodiac), *Madhyagraha* (Mean planets), *Suryasphuta* (True Sun), *Grahasphuta* (True planets), *Bhu-Vayu-Bhagola* (Spheres of the Earth, Atmosphere and Asterisms), *Ayanacalana* (Precession of the Equinoxes), *Pancadasaprasna* (Fifteen problems relating to spherical triangles), *Digjnana* (Orientation), *Chayaganita* (Shadow computations), *Lagna* (Rising point of the Ecliptic), *Nati-Lambana* (Parallaxes of Latitude and Longitude), *Grahana* (Eclipse), *Vyatipata,* and *Sringonnati* (Elevation of the Moon's horns).

Kriyakramakari, Yuktidipika and Yuktibhasha

There are two extensive commentaries, both by Sankara Variyar of Trikkutaveli family (AD 1500 – 1560), called *Kriyakramakari* and *Yuktidipika,* the former on the *Lilavati* of Bhaskara II, and the latter on the *Tantrasangraha* of Nilakantha Somayaji. Interestingly, there is a close affinity between the *Yuktibhasha* and the above said two commentaries. Even more, there are same sequence of arguments and verbal correspondences amongst them in the treatment of identical topics. From

this similitude it has been suggested that the *Yuktibhasha* is just a rendering into Malayalam of certain passages from these Sanskrit works. It is further suggested that for this reason, there is not much that is original in the *Yuktibhasha*. But the fact is just the other way round, namely that the Sanskrit versions are adaptations and paraphrases of the relevant portions from the *Yuktibhasha*. This is confirmed by Sankara, the author of both the commentaries, *Yuktidipika* and *Laghuvivritti* on *Tantrasangraha*, that what he had done in the commentary was only the setting out of the material elucidated in the work of the Brahmana of Parakroda (viz., Jyesthadeva, author of the *Yuktibhasha*). Cf., for instance one such colophonic verse:

इत्येषा परक्रोडावासद्विजवरसमीरितो योऽर्थः।
स तु तन्त्रसंग्रहस्य प्रथमेऽध्याये मया कथितः॥

Yuktibhasha and Ganita-Yuktibhasha

There is a work entitled *Ganita-Yuktibhasha* (Ms. No. R.4382 of the Government Oriental Manuscript Library, Chennai) in Sanskrit and it has been suggested that it might be the source of the Malayalam *Yuktibhasha*. However, a detailed comparison of the two shows that the *Ganita-Yuktibhasha* is but a rough and ready translation into Sanskrit of the Malayalam original by one who lacked not only the ability of writing idiomatic Sanskrit but also an adequate knowledge of the subject. Moreover, at places there occur haplographical omissions in the Sanskrit version of passages available in the Malayalam work, which fact too confirms that the Sanskrit version is the derived form.

The mathematical and astronomical rationale presented in the *Yuktibhasha* relate to several aspects, to wit, concepts, theories, constants, computations, demonstration by diagrammatic representation and the like. The procedure of depiction is logical, going step by step, first presenting the fundamentals and gradually building up the argument. It is, if one might say so, 'intimate' in that it includes the rationale and elucidates the steps even as a teacher does to a student. The work aims at understanding and conviction by the reader.

Conclusion

The account of Nilakantha Somayaji and his work as depicted above is a pointer not only to the work in the field of astronomy done by him, but also to the development of this discipline in this period in India.

Planetary Models in Indian and Greek Astronomical Traditions

M.S. Sriram

1. Introduction

In this article, we summarise the basic features of planetary models in Indian and Greek astronomical traditions, and compare them with each other. We shall also compare them with the Kepler's model, which is perhaps the last great work in the kinematical tradition, and correct in its essentials even today. It is also our aim to provide the necessary background to appreciate the innovations in the Indian planetary theory made by Nilakantha Somasutvan in his *Tantrasangraha* and other works, which will be discussed in the articles by K. Ramasubramanian on Nilakantha's planetary model and M.D.Srinivas on Nilakantha's geometrical picture of planetary motion.

The modern approach to planetary motions is of course dynamical and for completeness, we derive Kepler's model from Newton's law of gravitation in the second section. In section 3, we discuss the geocentric longitude and latitude of a planet in Kepler's model in a form convenient for comparison with ancient astronomy. In section 4, we outline the essentials of Ptolemy's model [1,3], and point out its inadequacies vis-a-vis Kepler's model. In section 5, we discuss the computation of geocentric longitude and latitude in the traditional Indian model (before Nilakantha) [5-9], and compare it with Ptolemy's as well as Kepler's models. Section 6 is a brief introduction to the planetary model of Copernicus [10]. Contrary to popular belief, Copernicus model is also fairly complicated as it is mainly a reformulation of Ptolemy's model for a heliocentric frame of reference, and differs substantially from Kepler's model. In sections 4, 5 and 6, we have emphasised the problems of equation of centre of interior planets and also latitude theory to enable the reader to appreciate Nilakantha's contributions discussed in the succeeding articles.

2. Planetary motion in modern astronomy and Kepler's laws.

In modern astronomy, the motion of a planet in the solar system is governed by Newton's law of gravitation, according to which the attractive gravitational force between any two bodies is proportional to the product of their masses and inversely proportional to the square of the distance between them. We ignore the interplanetary gravitational forces, as the Sun is far heavier than the planets. Then, we have to consider the force on a planet due to the Sun only. In the following, we shall employ the vector notation. In Fig.1, $\vec{r}_1$ and $\vec{r}_2$ refer to the position vectors of the Sun and the planet with masses M and m respectively.

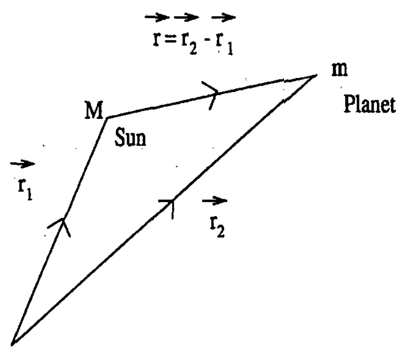

Fig. 1

The equations of motion of the planet and the Sun are given by

$$m \frac{d^2\vec{r}_2}{dt^2} = \frac{-GMm}{r^3} (\vec{r}_2 - \vec{r}_1), \tag{1}$$

$$M \frac{d^2\vec{r}_1}{dt^2} = \frac{GMm}{r^3} (\vec{r}_2 - \vec{r}_1), \tag{2}$$

where t is the time and G is Newton's gravitational constant. Subtracting Eq.(2) from Eq.(1), we find,

$$\frac{d^2\vec{r}}{dt^2} = \frac{-\mu}{r^3} \vec{r}, \tag{3}$$

where $\vec{r} = \vec{r}_2 - \vec{r}_1$ is the relative position vector from the Sun to the planet, and

$$\mu = G(M+m). \tag{4}$$

From Eq.(3), it is easy to show that

$$\frac{d}{dt}\left(\vec{r} \times \dot{\vec{r}}\right) = 0, \tag{5}$$

where $\dot{\vec{r}} = \frac{d\vec{r}}{dt}$ is the velocity vector (overdot refers to time-derivative). This implies that the motion of the planet is in a plane and that the 'areal velocity' (the area swept by the radial line connecting the Sun and the planet per second) is constant.

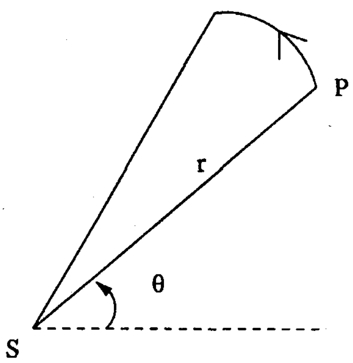

Fig. 2

In the polar coordinates shown in Fig.2, this implies that

$$\text{Areal velocity} = \tfrac{1}{2}\left|\vec{r} \times \dot{\vec{r}}\right| = \tfrac{1}{2} r^2 \dot{\theta} = \text{Constant} = h. \tag{6}$$

Also, the radial component of Eq.(3) gives

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2}. \tag{7}$$

Using Eq.(6), we have

$$\ddot{r} - \frac{4h^2}{r^3} = \frac{-\mu}{r^2}. \tag{8}$$

The solution of this second order differential equation in r is given by

$$r = \frac{l}{1 - e\cos(\theta - \varpi)} \tag{9}$$

where $l = \dfrac{h^2}{\mu}$, while e and ϖ are constants of integration.

This is the equation of an ellipse with e as the eccentricity. ϖ is the angle between the major axis of the ellipse and the reference axis. Hence, application of Newton's law of gravitation to the motion of planets gives us the following results.

1. Each planet moves around the Sun in an ellipse, with Sun as one of the foci.

2. The areal velocity of a planet in its orbit is a constant.

Further we can show that if T_1, T_2 are the time periods of two planets 1 and 2, and a_1, a_2 their semi major axes,

$$\frac{T_2^2}{T_1^2} = \frac{(M + m_1)a_2^3}{(M + m_2)a_1^3}. \tag{10}$$

When $M >> m_1$, m_2, this implies that

$$3. \quad \frac{T_2^2}{T_1^2} = \frac{a_2^3}{a_1^3}, \text{ or, } T^2 \propto a^3. \tag{11}$$

(1), (2) and (3) are nothing but Kepler's laws. Hence, Kepler's laws for planetary orbits can be derived from Newton's law of gravitation. We should note that Kepler's laws are essentially kinematical laws, which do not make any reference to the concepts of 'acceleration' and 'force', as we understand them. Even then, they capture the very essence of the nature of the planetary orbits and can be used to calculate the planetary positions, once we know the parameters of the ellipse and the initial positions. The planetary models in ancient astronomy are kinematical and should be really compared with Kepler's model. The planetary models in Indian or Greek traditions give procedures or explicit geometrical models to calculate the geocentric longitude and latitude of a planet. In the following, we elaborate on the computation of the geocentric longitude and latitude of a planet in Kepler's model, so that the ancient Indian and Greek models can be compared with it.

3. Geocentric longitude and latitude of a planet in Kepler's model

(a) Elliptic orbits and equation of centre

The elliptic orbit of a planet (P) around the Sun (S) is represented in Fig. 3. a and b are the semi-major and semi-minor axes of the ellipse.

Υ refers to the first point of Aries. ϖ is the longitude of the aphelion (A). $\theta_h = \Upsilon \hat{S} P$ is the heliocentric longitude of the planet. From Eq. (9),

$$\frac{l}{r} = 1 - e\cos(\theta_h - \varpi) \tag{12}$$

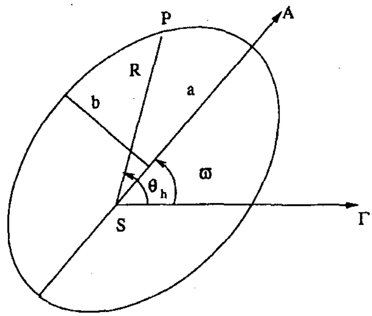

Fig. 3: Elliptic orbit of a planet P around the Sun.

Therefore,

$$r = l[1 + e\cos(\theta_h - \varpi)] + O(e^2), \tag{13}$$

$$r^2 = l^2[1 + e\cos(\theta_h - \varpi)]^2 + O(e^2)$$
$$= l^2[1 + 2e\cos(\theta_h - \varpi)] + O(e^2). \tag{14}$$

Now the area of the ellipse is πab. Hence the areal velocity, which is a constant is $\dfrac{\pi ab}{T} = \dfrac{\omega ab}{2}$, where T is the time period and ω is the mean angular velocity of the planet. Using Eq.(6) we have,

$$r^2\dot{\theta}_h = \omega ab. \tag{15}$$

Hence,

$$l^2\dot{\theta}_h[1 + 2e\cos(\theta_h - \varpi)] = \omega ab + O(e^2), \tag{16}$$

using Eqs.(14) and (15). Now

$$l^2 = a^2(1 - e^2) = a^2 + O(e^2) \text{ and } ab = a^2 + O(e^2). \tag{17}$$

Hence,

$$\dot{\theta}_h[1 + 2e\cos(\theta_h - \varpi)] \approx \omega, \tag{18}$$

where the equation is correct to $O(e)$. Integrating this with respect to time, we find

$$\theta_h + 2e\sin(\theta_h - \varpi) \approx \omega t, \tag{19}$$

or, again to $O(e)$

$$\theta_h - \theta_M = \theta_h - \omega t \approx -2e\sin(\omega t - \varpi) \tag{20}$$

$$= -2e\sin(\theta_M - \varpi).$$

Here $\theta_M = \omega t$ is the mean longitude which increases linearly with time t. $\omega t - \varpi$, that is the difference between the longitudes of the mean planet and the apogee is the 'anomaly'. Eq.(20) gives the equation of centre which is the difference between the true heliocentric longitude θ_h and the mean longitude θ_M, to $O(e)$, in terms of the anomaly. Clearly, the equation of centre is a consequence of the eccentricity of the orbit.

(b) Geocentric longitude of an exterior planet

The orbits of all the planets are inclined at small angles to the plane of Earth's orbit around the Sun. We will ignore these inclinations and assume that all the planetary orbits lie in the ecliptic plane for the calculation of planetary longitudes, as the correc-tions introduced by them (inclinations) are known to be small. We will consider the longitude of an exterior planet like Mars, Jupiter or Saturn and an interior planet like Mercury or Venus, separately.

The elliptic orbit of an exterior planet (P) and that of the Earth (E) around the Sun are shown in Fig. 4. Here, $\theta_h = \Upsilon\hat{S}P$ is the true heliocentric longitude of the planet. $\theta_s = \Upsilon\hat{E}S$ *and* $\theta_g = \Upsilon\hat{E}P$ are the true geocentric longitudes of the Sun and the planet respectively, while r and R are the distances of the Earth and the planet from the Sun, which vary along their orbits. To facilitate comparison with the ancient planetary models, $EP' = R$ is drawn parallel to SP. Then $P'P$ is parallel to ES and $P'P = r$. We have already described how θ_h is computed using the expression for the equation of centre. From this, the true geocentric longitude, θ_g has to computed.

Now

$$E\hat{P}S = P\hat{E}P' = \theta_g - \theta_h, \tag{21}$$

and

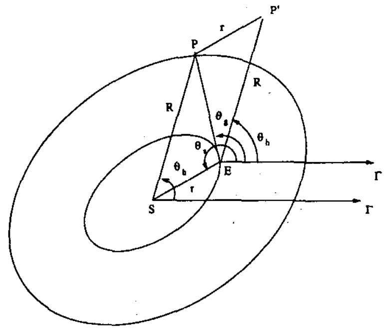

Fig. 4: Heliocentric and geocentric longitudes of an exterior planet in Kepler's model.

$$E\hat{S}P = 180^0 - P'\hat{E}S = 180^0 - (\theta_s - \theta_h). \tag{22}$$

In the triangle ESP,

$$EP^2 = R^2 + r^2 - 2rR\cos\left[180^o - (\theta_s - \theta_h)\right]$$

$$= R^2 + r^2 + 2rR\cos(\theta_s - \theta_h)$$

$$= \left[R + r\cos(\theta_s - \theta_h)\right]^2 + r^2\sin^2(\theta_s - \theta_h),$$

or,

$$EP = \left[\{R + r\cos(\theta_s - \theta_h)\}^2 + r^2\sin^2(\theta_s - \theta_h)\right]^{1/2} \tag{23}$$

Also

$$\frac{\sin(E\hat{P}S)}{ES} = \frac{\sin(E\hat{S}P)}{EP}. \tag{24}$$

Using Eqs. (21) – (24)

$$\sin(\theta_g - \theta_h) = \frac{r\sin(\theta_s - \theta_h)}{\left(\{R + r\cos(\theta_s - \theta_h)\}^2 + r^2\sin^2(\theta_s - \theta_h)\right)^{1/2}}. \tag{25}$$

$\theta_s - \theta_h$, the difference between the longitudes of the Sun and the heliocentric planet is the 'solar anomaly'. Eq.(25) gives $\theta_g - \theta_h$ in terms of the solar anomaly. Adding this to θ_h, we get the true geocentric longitude, θ_g of the planet.

Note that the true planet P can be located in another manner. Locate

P' which has the heliocentric longitude but with respect to the Earth and which is at the same distance from the Earth, as the true planet is from the Sun. Locate P such that it is at a distance ES from P' and $P'P$ is parallel to ES. In other words, the elliptic orbit of the planet around the Sun can be replaced by an identical orbit around the Earth and the apparent elliptic orbit of the Sun around the Earth can be replaced by an identical orbit around the 'planet' P' to find the geocentric location of the planet. This is the principle behind the computation of the geocentric longitude of an exterior planet in ancient astronomical models. It can be summarised in the simple vector equation,

$$\overrightarrow{ES} + \overrightarrow{SP} = \overrightarrow{EP'} + \overrightarrow{P'P}. \tag{26}$$

(c) Geocentric longitude of an interior planet.

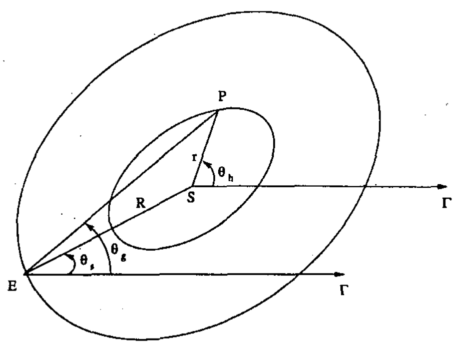

Fig. 5: Heliocentric and geocentric longitudes of an interior planet in Kepler's model.

The elliptic orbits of an interior planet (P) and that of the Earth (E) around the Sun are shown in Fig.5. $\theta_h = \Upsilon\hat{S}P$ is the true heliocentric longitude of the planet, which can be computed from the mean heliocentric longitude and the equation of centre. $\theta_s = \Upsilon\hat{E}S$, and $\theta_g = \Upsilon\hat{E}P$ are the true geocentric longitudes of the Sun and the planet, respectively. Here r and R are the variable distances of the planet and the Earth from the Sun, respectively. Note the change in nomenclature, compared with the one for an exterior planet.

Now,

$$S\hat{E}P = \theta_g - \theta_s. \tag{27}$$

It can be easily seen that

$$E\hat{S}P = 180^0 - (\theta_h - \theta_s), \tag{28}$$

$$EP = \left[\{R + r\cos(\theta_h - \theta_s)\}^2 + r^2 \sin^2(\theta_h - \theta_s)\right]^{1/2}. \tag{29}$$

Also,

$$\frac{\sin(S\hat{E}P)}{SP} = \frac{\sin(E\hat{S}P)}{EP}. \tag{30}$$

Using Eqs. (27) – (30),

$$\sin(\theta_g - \theta_s) = \frac{r\sin(\theta_h - \theta_s)}{\left(\{R + r\cos(\theta_h - \theta_s)\}^2 + r^2\sin^2(\theta_h - \theta_s)\right)^{1/2}}. \tag{31}$$

$\theta_g - \theta_s$ is determined from this equation. Adding this to θ_s, we get the true geocentric longitude θ_g of the planet. Note that the true longitude of the Sun, θ_s, and the true heliocentric longitude of the planet, θ_h (obtained by adding the equation of centre for the planet to the mean heliocentric longitude), should be obtained first. Then the true geocentric interior planet can be obtained using Eq.(31). For an interior planet, the apparent elliptic orbit of the Sun around the Earth and the elliptic orbit of the planet around the Sun should be used to locate the true geocentric planet, $\overrightarrow{EP}$.

(d) Heliocentric and geocentric latitudes of a planet

In Fig.6, the orbit of the planet *P* is inclined at an angle i_h to the ecliptic. N and *N'* are the nodes. *PP'* is the circular arc perpendicular to the ecliptic. Then the heliocentric latitude β_S is given by

$$\beta_S = P\hat{S}P' = \frac{PP'}{SP}. \tag{32}$$

If λ_P and λ_N are the heliocentric longitudes of the planet and the node, it can be shown that

$$\sin\beta_S = \sin i_h \sin(\lambda_P - \lambda_N), \tag{33}$$

or

$$\beta_S \approx i_h \sin(\lambda_P - \lambda_N), \tag{34}$$

as i_h and β_S are small. Note that λ_P stands for the true heliocentric longitude, whether it is an exterior or an interior planet. We have also

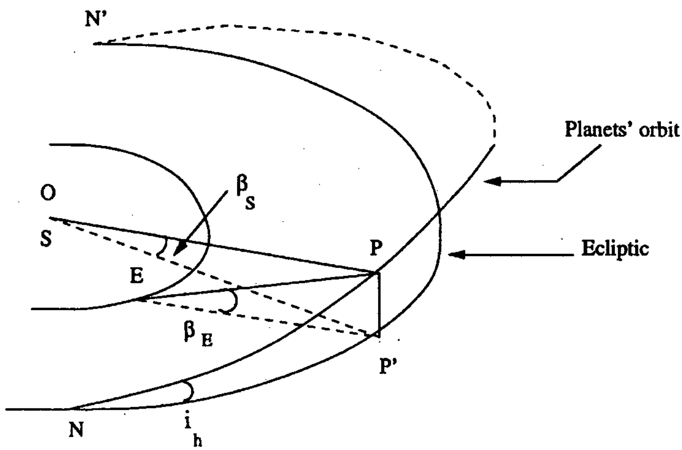

Fig. 6: Heliocentric and geocentric latitudes of a planet in Kepler's model.

shown the Earth's orbit in the figure. The geocentric latitude β_E is given by

$$\beta_E = P\hat{E}P' = \frac{PP'}{EP} \ . \tag{35}$$

From Eqs.(32) - (35), we find that

$$\beta_E = \beta_S \frac{SP}{EP}$$

$$= \frac{i_h \ SP \sin(\lambda_P - \lambda_N)}{EP} \tag{36}$$

where EP, the true distance of the planet from the Earth, can be found from Eq.(23) or Eq.(29).

4. Planetary models in Greek astronomy : Ptolemy's model

Long before Ptolemy, it had been noticed that the motions of the Sun, Moon and the planets Mercury, Venus, Mars, Jupiter and Saturn (which are visible) in the back-ground of stars were not regular. Appolonius of Perga (230 B.C.) is credited with the epicycle or eccentric models to

account for the irregularities. Hipparchus (130 BC) was able to explain the solar motion fairly successfully, using the epicycle or the eccentric models (which were shown to be equivalent) to account for the *equation of centre*. It was left to Ptolemy (c.150 AD) to give a more satisfactory account of Moon's motion, with the aid of a second *irregularity*, the *evection*. As regards the motion of the five planets, there was little headway before Ptolemy. In his Mathematical Syntaxis or *Almagest*, Ptolemy explicity states that he was the first to establish a theory of the planetary motions. We give a brief account of Ptolemy's planetary model, in the following [1,3]. We first describe the epicycle and eccentric circle models, as they are the essential ingredients of the full model.

(a) Epicycle and eccentric circle models.

First the epicycle model. Let *ABCD* be a circle of radius *R* in the plane of the ecliptic, with the Earth *E*, as the centre. This is the *deferent circle*. *A* represents the direction of the apogee, that is, $\Upsilon\hat{E}A = \varpi$. The mean planet *M* moves along *ABCD* uniformly, that is, $\theta_0 = \Upsilon\hat{E}M = \omega t$. Imagine a circle of radius *r* with *M* as the centre. This is the epicycle (see Fig. 7). The true planet *P* moves on this epicycle, in the direction opposite to *M*, but at the same rate. Further when *M* is at *A*, *P* is directly above it at P_0. When *M* is at an arbitrary position, extend *EM* to meet the epicycle at *Q* and draw *PF* perpendicular to *EQ*.

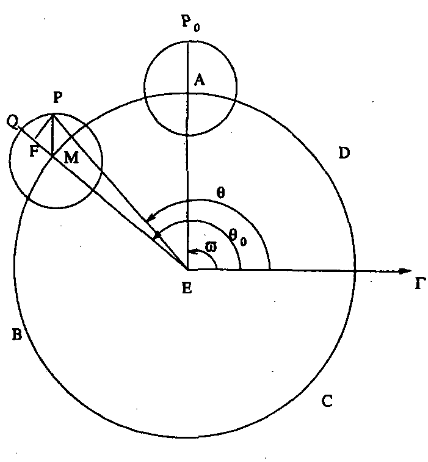

Fig. 7: The epicycle model

Then the true planet is at P, such that $Q\hat{M}P = M\hat{E}A = \theta_0 - \varpi$. Hence the true longitude of P, θ or $\Upsilon\hat{E}P$, is given by,

$$\theta = \Upsilon\hat{E}P = \theta_0 - P\hat{E}Q. \tag{37}$$

Now

$$\sin(P\hat{E}Q) = \frac{PF}{EP} = \frac{r\sin(Q\hat{M}P)}{EP} = \frac{r\sin(\theta_0 - \varpi)}{EP}. \tag{38}$$

We have

$$\begin{aligned} EP^2 &= EF^2 + FP^2 \\ &= (EM + MF)^2 + FP^2 \\ &= (R + r\cos(\theta_0 - \varpi))^2 + r^2\sin^2(\theta_0 - \varpi). \end{aligned} \tag{39}$$

Hence,

$$\sin(\theta_0 - \theta) = \frac{r\sin(\theta_0 - \varpi)}{\left(\{R + r\cos(\theta_0 - \varpi)\}^2 + r^2\sin^2(\theta_0 - \varpi)\right)^{1/2}}. \tag{40}$$

If $\frac{r}{R} << 1$,

$$\sin(P\hat{E}Q) \approx P\hat{E}Q \text{ and } EP \approx R. \tag{41}$$

Then,

$$P\hat{E}Q \approx \frac{r}{R}\sin(\theta_0 - \varpi) = \frac{r}{R}\sin(\omega t - \varpi), \tag{42}$$

and

$$\theta - \theta_0 \approx -\frac{r}{R}\sin(\omega t - \varpi). \tag{43}$$

This is the expression for the equation of centre in the epicycle model, in radians, when $r/R << 1$. This is the same as Eq.(20) in Kepler's model, if we identify r/R with $2e$. Hence, the equation of centre in Kepler's model is reproduced in the epicycle model to the first order in eccentricity.

In the eccentric circle model, the true planet moves uniformly with the same rate of motion as the mean planet, but on a circle of radius R with its centre O on the line EA, such that $EO = r$. The mean longitude θ_0 and the geocentric longitude θ are indicated in Fig.8. It is easy to see that the location of P with respect to E is the same in both the epicycle and the eccentric models, by comparing Figs.7 and 8. Hence, the models are equivalent and give the same result for the equation of centre.

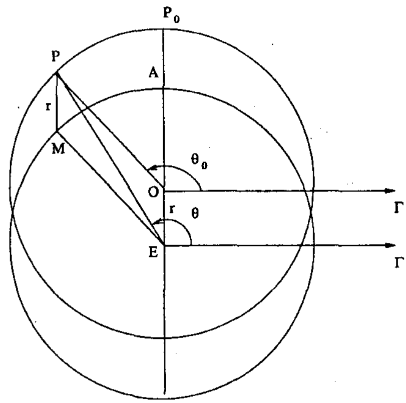

Fig. 8: The eccentric circle model

(b) Longitudes of planets in Ptolemy's model

Ptolemy accepts the order of the planets assumed by the ancients, namely, Moon, Mercury, Venus, Sun, Mars, Jupiter and Saturn from the Earth upwards. The central tenet of Ptolemy's system is that all irregularities in the motion of a planet can be accounted for through combinations of uniform and circular motions. Ptolemy states that there are two apparent anomalies for each of the five planets: (i) the *zodiacal anomaly* which depends upon the position of the planet on the ecliptic (essentially the equation of centre), and (ii) the *solar anomaly*, which varies according to its position relative to the Sun (to convert the heliocentric to geocentric longitudes, in the modern perspective).

Ptolemy gives the mean periods and the synodic periods (the difference between the periods of the mean Sun and the mean heliocentric planet). For the interior planets, the mean Sun is the mean planet. Ptolemy's model for Mercury is slightly different from that of the other planets. We will take up the later first. Ptolemy combines the eccentric and epicyclic systems, the former to account for the *zodiacal anomaly* and the later for the *solar anomaly*.

Let *ABC* the deferent circle of radius *R* concentric with the ecliptic with *E*, the centre of the Earth, as centre. *A* and *C* are the apogee and perigee of the planet, respectively, that is, $\Upsilon\hat{E}A = \varpi$, the longitude of the apogee. The centre of the epicycle moves on an eccentric circle *FGH*, with the same radius as the circle *ABC*, but whose centre is *D*, which lies

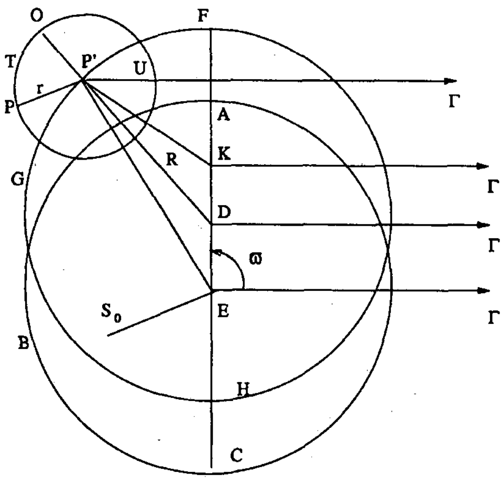

Fig. 9: Ptolemy's model for longitudes

on the diameter AE. K is a point on AD, such that $KD = DE$. K is called the *Equant*. The mean planet P' moves on the circle FGH, uniformly about K (not D, the centre of the eccentric circle). Therefore, $\Upsilon\hat{K}P' = \theta_0 = \omega t$. With P' as the centre, draw an epicycle of radius r, OTU. Then, the planet P moves uniformly along the epicycle, such that it completes its revolution with respect to the diameter directed towards K, in a period equal to that of the synodic period. Hence, when the planet is at P, $O\hat{P}'P =$ Mean Sun - Mean planet , or $\Upsilon\hat{P}'P =$ Mean Sun, as $\Upsilon\hat{P}'O =$ Mean planet, for an exterior planet. This means that $P'P$ is parallel to ES_0, where S_o denotes the mean Sun. For Venus, $\Upsilon\hat{P}'P =$ Mean heliocentric planet, and $\Upsilon\hat{P}'O =$ Mean Sun. For calculating the longitude above, the planes of the eccentric and the epicycle are considered to be the same as that of the ecliptic, because no considerable difference in longitude arises because of the inclinations of these planes to the ecliptic. Finally, the geocentric planet is P and the geocentric longitude of the planet is $\Upsilon\hat{E}P$.

We now compare the geocentric longitudes of the planet in the models of Ptolemy and Kepler. Ptolemy does not use the simple eccentric circle model to compute the *zodiacal anomaly* or the *equation of centre*, as the centre of the epicycle P' moves uniformly about K, the *equant* which is not the centre of the eccentric circle. Even then it can account for the

correct equation of centre, to the first order in eccentricity, by choosing $\frac{ED}{R}$ appropriately.

We first consider the exterior planets. EP' would have the same significance as in Fig. 4 for Kepler's model, so that $\gamma\hat{E}P'$ would give the true heliocentric planet θ_h, to this order. Similarly, the epicycle would correspond to the apparent orbit of the mean Sun around the Earth. The expression for θ_g, the true geocentric longitude of the planet would be the same as in Kepler's model, given by Eq.(25) with θ_s replaced by θ_{S_0}, the mean Sun, provided that r/R in Ptolemy's model is the same as the ratio of the Earth - Sun and planet-Sun distances. In Table 1, we compare the ratio r/R with the ratio of the mean distances, as per modern astronomy and find that it is indeed true. Hence Ptolemy's model for the longitudes of the exterior planets is basically correct, but for the fact that mean Sun is used in the model, instead of the true Sun, as it should have been.

The story is different for Venus. Comparing with Fig.5, we see that P' should have been the true Sun, that is mean Sun corrected for its equation of centre and P should have been the true heliocentric planet, that is the mean heliocentric planet corrected for its equation of centre. Neither of this is true in Ptolemy's model. With respect to E, P' is the mean Sun corrected for the equation of centre of the planet. With respect to P', P is the mean heliocentric planet, uncorrected for the eccentricity of its orbit, as $\gamma\hat{P}'P =$ mean heliocentric planet. Though we can recover the expression for θ_g in Eq.(31), θ_h and θ_s would be wrong. Hence, the formulation of the equation of centre for Venus is wrong in Ptolemy's model.

The model for Mercury should have been no different from the one for Venus. In the following, we give Ptolemy's model for Mercury, which is more complicated.

Let ABC be a circle with centre as E, the centre of the Earth and in the plane of the ecliptic. A and C are the apogee and perigee respectively. D and K are points on EA such that $ED = DK$. Here, D is not the centre of the eccentric circle as for the other planets. Rather, the centre of the eccentric circle L, carrying the epicycle moves about K westwards (that is, in the direction opposite to that of the mean motion), always at a distance KD from K. With L as the centre, draw a circle MNO equal to the circle ABC. The centre of the epicycle P' moves uniformly around D. The centre of the eccentric circle L moves on a circle, uniformly about K

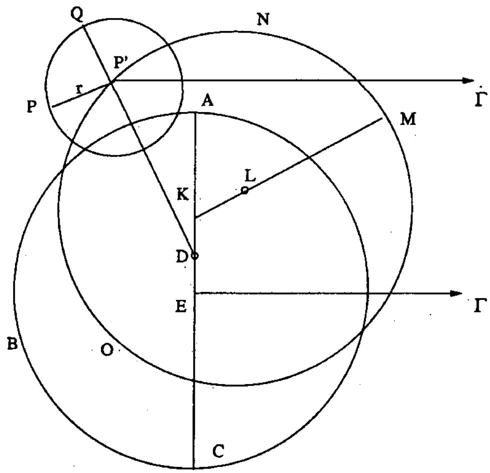

Fig. 10: Ptolemy's model for Mercury.

westward and at the same rate as that of P', the centre of the epicycle, eastward. The true planet P moves on the epicycle such that it completes its revolution with respect to the diameter directed towards D, in a period equal to that of the synodic period. As in the case of Venus, $\Upsilon\hat{P'}P$ = mean heliocentric planet and $\Upsilon\hat{E}P$ is the true geocentric longitude.

Ptolemy could not explain the observations on Mercury's longitude using a model with a fixed eccentric circle. If the centre of the eccentric circle is not fixed, it has to be moving uniformly on a circle! Its motion was taken equal and opposite to that of the mean planet, to account for the observations of the greatest elongations of Mercury. Of course, this motion in Ptolemy's model for Mercury has no equivalent in Kepler's model.

Planet	Ptolemy's ratio	Modern value
Mercury	0.371	0.387
Venus	0.719	0.723
Mars	0.658	0.656
Jupiter	0.192	0.192
Saturn	0.108	0.105

Table 1: Comparison of r/R, ratio of the radii of the epicycle and the deferent circle in Ptolemy's model with the modern values of r/R (ratio of the mean values of the Earth-Sun and Planet-Sun distances for exterior planets and the inverse ratio for interior planets).

(c) Ptolemy's model for planetary latitudes

We give a brief account of Ptolemy's model for planetary latitudes, in the following. Its central feature is that the line of nodes of the inclined planetary planes pass through the Earth, rather than the Sun.

For an exterior planet, the deferent and the eccentric circles were inclined to the ecliptic at a fixed angle. The epicycle was inclined to the plane of the eccentric, such that the diameter perpendicular to the diameter through the apogee of the epicycle was parallel to the plane of the ecliptic. In his major work *Almagest*, Ptolemy proposes that the inclination of the epicycle to the eccentric varied when the centre of the epicycle moved along it. Near the apogee and perigee of the deferent, the inclination had the maximum value. The epicycle was in the plane of the ecliptic when its centre was at one of the nodes. Ptolemy states that he was led to this assumption by his observation that at the apogee and perigee of the deferent, the latitude was greatest when the planet happened to be at the perigee of the epicycle. In his later work, *Planetary Hypothesis*, Ptolemy gives a better model in which the epicycle is inclined at the same fixed angle to the eccentric, as the eccentric is inclined to the ecliptic, such that its plane is always parallel to the ecliptic. This is of course correct, as the epicycle represents the apparent orbit of the Sun around the Earth. Still the inclination of the ecliptic to the eccentric was unusual in the Greek framework and Ptolemy had to explain it by introducing an auxiliary circle.

The interior planets, Mercury and Venus were treated diferently. In Fig.11, *A* is the apogee and *P*, the perigee of the deferent. *NN'* is the line of nodes or the line of intersection of the planes of the deferent and ecliptic, which passes through the Earth. *acbd* is the epicycle, with *ab* as the line of apsides and *cd*, the diameter perpendicular to it. When the epicycle is at *A* or *P*, *ab* lies in the plane of the deferent, while *cd* is inclined to it at an angle called the *obliquatio*. At the nodes *N* and *N'* the diameter *cd* lies in the plane of the ecliptic and *ab* is inclined to the deferent. This tilting of the epicycle is called *inclination*. The plane of the deferent oscillates within some limit to both sides of the ecliptic. When the epicycle of Venus is at the ascending node *N'*, the inclination of the deferent is zero. As the epicycle advances, the point *A* rises north of the ecliptic and continues to rise till the epicycle reaches *A*. After that the latitude decreases until it becomes zero, but thereafter the part *NPN'* rises north of the ecliptic carrying the epicycle with it. Hence, the centre of the epicycle always has a northern latitude, except at *N* and *N'*. Simultaneously, the double rocking

of the epicycle is going on. For Mercury, the theory is similar, except for the fact that the point A is always south of the ecliptic, instead of north.

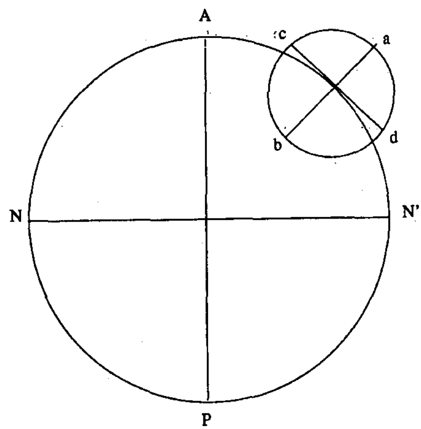

Fig. 11: Ptolemy's model for latitudes of Mercury and Venus

So, the latitude theory is fairly complicated, as confessed by Ptolemy himself. In fact he is very apologetic about it and acknowledges his inability to formulate a simple model in the following words [2]:

> Now let no one, considering the complicated nature of our devices, judge such hypotheses to be over-elaborated. For it is not appropriate to compare human (construction) with divine, nor to form one's beliefs about such great things on the basis of very dissimilar analogies. For what (could one compare) more dissimilar than the eternal and unchanging with the ever-changing, or that which can be hindered by anything with that which cannot be hindered even by itself? Rather, one should try, as for as possible, to fit the simpler hypotheses to the heavenly motions, but if this does not succeed, (one should apply hypotheses) which do fit

That Ptolemy found the latitudes of the planets extremely troublesome is not surprising, considering the fact that he assumed that their lines of nodes pass through the Earth, whereas they pass through the Sun, in reality. In the words of Dreyer, the well known historian of astronomy [4],

> In no other part of planetary theory did the fundamental error of the Ptolemaic system cause so much difficulty as in accounting for the latitudes, and these remained the chief stumbling block up to the time of Kepler.

There is another aspect of Ptolemy's model which needs to be emphasised. Using the deferent, eccentric circle and the epicycles, the position of the planet is no doubt located on the epicycle. However, the calculation of the geocentric longitude was no simple matter for Ptolemy as it involves the sine and cosine functions (see Eqs.(20), (25) and (31)), which were not known to him. So, there are no explicit trigonometrical expressions relating the true longitude and latitude to the mean longitude, and the 'anomalies'. He had to use detailed geometrical arguments and take recourse to tables relating the chords and arcs of a circle, to find the true geocentric position of a planet. This is cumbersome, though not wanting in accuracy.

5. Planetary models in Indian astronomy

Indian astronomical texts or *siddhantas* give explicit procedures for computing the position of a planet. According to tradition, there were 18 *siddhantas* which were prior to Aryabhata. However, authentic versions of these ancient *siddhantas* are not available to us, and it is in *Aryabhatiya* (499 AD) of Aryabhata [5] that we find a complete account of the procedures for calculating the positions of the Sun, Moon and planets, diurnal problems, eclipses, etc., for the first time. The essential approach is the same in the later *siddhantas*, though they may elaborate on the procedures as *Aryabhatiya* is cryptic; they may also differ in details. We saw that in the Greek tradition, the emphasis is on an explicit geometrical model for planetary motion. In Indian *siddhantas*, the emphasis is more on the computational procedures for the geocentric longitude and latitude of a planet in terms of the mean longitude and the anomalies, directly. This was possible as they were familiar with the sine and cosine of an angle. However, geometrical models underlying the computations are to be found in some of the commentaries.

The basic framework is geocentric, with the Earth as the centre. In essence, the computation of the geocentric longitude of a planet proceeds in three steps [5-9]. First, the mean longitude is computed. Next, a correction is applied to the mean planet to account for the non-uniformity in motion or the anomaly pertaining to the apogee. This is the *mandasamskara.* Finally, the *sighrasamskara* is carried out to obtain the true geocentric longitude. This is the correction due to *Solar anomaly.* We explain each of these steps in the following.

(a) Mean longitude

From Aryabhata onwards, one considers a *Mahāyuga* of 43,20,000 years. The number of revolutions in the stellar background made by the planets in this *Mahayuga* is given in the texts. The number of civil days in a *Mahayuga* or *yugasavanadina* is also given. For instance, the *yugasavanadina* is 1577917500 in *Aryabhatiya*. The number of revolutions of the planets in a *Mahayuga* according to this text are reproduced in Table 2 and the periods of revolution are compared with the modern values. It is to be noted that for Mercury and Venus, the values given are those of their *sighrocchas* (mean heliocentric planet, in modern terminology).

Planet	Number of Revolutions in a *Mahayuga*	Period of revolution in days	
		Aryabhatiya	Modern value
Sun	43,20,000	365.25868	365.25636
Moon	5,77,53,336	27.32167	27.32166
Moon's apogee	4,88,219	3231.98708	3232.37543
Moon's nodes	2,32,226	6794.74951	6793.39108
Mercury *sighroccha*	1,79,37,020	87.96988	87.96930
Venus *sighroccha*	70,22,388	224.69814	224.70080
Mars	22,96,824	686.99974	686.97970
Jupiter	3,64,224	4332.27217	4332.58870
Saturn	1,46,564	10766.06465	10759.20100

Table 2: Planetary revolutions in a *Mahayuga*, according to *Aryabhatiya*. *Yugasavanadina* = 1577917500.

For finding the mean longitude at any time, the *ahargana* is computed first. This is the number of days elapsed since an epoch. For instance, the *Kali-Ahargana* is the number of days elapsed since the begining of *Kaliyuga*, which is the midnight of February 17-18, 3102 BC. As the calendar is luni-solar, a particular day is specified by the year, lunar month and *tithi*. Then *ahargana* has to be carefully computed taking into account the *adhikamasas* (extra lunar month in some lunar years). All the mean longitudes are assumed to be zero at the begining of *Kaliyuga*. Then, the mean longitude at a given instant can be readily computed after adding the time interval between the midnight and the desired instant to the *ahargana*, with the knowledge of the planet's period of revolution.

An epoch other than the beginning of the *Kaliyuga* is also often used

in the texts. In that case, the mean longitudes of the planets at that epoch are stated. Many texts also prescribe *dhruvakas* or zero corrections to the mean longitudes at the beginning of *Kaliyuga.*

(b) Manda Samskara

The *equation of centre* is calculated using either the epicycle model shown in Fig.7 or the eccentric model in Fig.8 . It was known that they were equivalent. The circumference of the deferent circle, known as *Kakshyamandala,* $2\pi R$ is taken to be 360°, normally. The mean planet is known as *Madhyamagraha* (θ_M) and the apogee as *Mandoccha* (ϖ). The angle between them is *Mandakendra* (anomaly) $= \theta_M - \varpi$. The circumference of the epicycle, $2\pi r$, is known as *Mandavrittaparidhi.* The *Mandakendrajya* is $R \sin$ (*Mandakendra*). The equation of centre is known as *Mandaphala* and the planet corrected for the equation of centre is known as *Mandasphutagraha* (θ_{MS}) . The *siddhantas* give the following formula for the *Mandaphala* (in degrees) :

$$\textit{Mandaphala} = \frac{-\textit{Mandavrittaparidhi} \times \textit{Mandakendrajya}}{360^0}.$$

This has to be added to the *Madhyamagraha* to obtain the *Mandasputagraha.* It is easily seen that

$$\begin{aligned} \textit{Mandaphala} &= \theta_{MS} - \theta_M \text{ (in degrees)} \\ &= \frac{-180^0 \times r}{\pi R} \sin(\theta_M - \varpi) \end{aligned} \tag{45}$$

as $2\pi R = 360^0$. Eq.(45) coincides with the equation of centre in Kepler's model to $O(e)$, Eq.(20), apart from a factor 180°/π to convert radians to degrees, when we identify θ_{MS} with the true heliocentric planet θ_h and r/R with 2e. The *Mandavrittaparidhi* and the *mandoccha* for each planet have been listed in the texts, so that the equation of centre can be computed directly.

In the Indian astronomy texts, there is a clear distinction between the exterior and the interior planets. For the exterior planets, Mars, Jupiter and Saturn, the mean planet θ_M is the same as the mean heliocentric planet and the *Mandasphutagraha* is the same as the true heliocentric planet. So the procedure is essentially correct. In the traditional Indian model (before Nilakantha Somayaji), the mean planet for the interior planets, Mercury and Venus is identified with the mean Sun, and the

equation of centre for the planet is added to it. The *Mandasphutagraha* thus obtained has no physical significance. So, it has the same defect as Ptolemy's model, as far as the equation of centre for the interior planets is concerned.

There is one important difference between the Indian and Greek methods. Whereas the radius of the epicycle of a planet, r is constant in the Greek approach, it is not so in the Indian system. Generally $r = r_0 - \alpha|\sin M|$, that is, it has a value r_0 at the end of an even quadrant and the value $r_0 - \alpha$ at the end of an odd quadrant. The values of r_0 and α depend upon the planet and vary from text to text. The motivation for introducing a variable radius of the epicycle is not clear. Recently, it has been demonstrated that it leads to a 'quasi-Keplerian model' with the following feature. The orbit is made up of two elliptical arcs corresponding to two ellipses whose major axes are inclined to the apseline in opposite senses and joined together[1].

(c) Sighra samskara

In the Indian texts, both the eccentric and epicycle models are used to calculate the *Solar anomaly.* This is in contrast to Ptolemy who explicitly states that only the epicycle model has to be used for this, though he could have as well used the other model. Fig.7 can also be used for *Sighrasamskara* with the following changes. *M* is now the *Mandasphutagraha*, whose longitude is $\Upsilon\hat{E}M = \theta_{MS}$, which has been computed in the previous subsection. Now *A* stands for the *sighroccha*, which is identified with the mean Sun for exterior planets and the mean heliocentric planet for interior planets. $\theta_{Si} = \Upsilon EA$ is the longitude of the *sighroccha.* The angle between *sighroccha* and the *Mandasphutagraha* $(\theta_{Si} - \theta_{MS})$ is known as *Sighrakendra. P is* identified with the true geocentric planet, so that $\Upsilon\hat{E}P = \theta_g$, is the true geocentric longitude of the planet. Replacing θ_0 by θ_{MS}, ϖ by θ_{Si} and θ by θ_g in Eq.(40), we find

$$\sin(\theta_g - \theta_{MS}) = \frac{r\sin(\theta_{Si} - \theta_{MS})}{\left(\{R + r\cos(\theta_{Si} - \theta_{MS})\}^2 + r^2\sin^2(\theta_{Si} - \theta_{MS})\right)^{1/2}} . \quad (46)$$

This is the formula for computing *Sighraphala*, $\theta_g - \theta_{MS}$, in the Indian texts, where the numerator and the denominator on the right hand side are known as *Dohphala* and *Sighrakarna*, respectively. Adding this to the *Mandasphutagraha* θ_{MS}, we obtain the true geocentric longitude

1. See the article by S. Madhavan in this volume.

θ_g The whole process is known as *Sighrasamskara*. Just as in *Mandasamskara*, the radius of the epicycle in *Sighrasamskara* is not a constant, but depends on the *Sighrakendra*. Different texts give different prescriptions for the radius R.

For the exterior planets, the *sighroccha* is identified with the mean Sun, so that $\theta_{Si} = \theta_{S_0}$, and θ_{MS} is the true heliocentric planet θ_h. Then Eq.(46) is nearly the same as Eq.(25) in Kepler's model, provided *r/R* in the Indian model is equal to the ratio of the Earth-Sun and planet-Sun distances. In Table 3, we compare the ratio *r/R* with the ratio of the mean distances, as per modern astronomy and find that this is largely true. *Hence, the Indian model for the longitudes of the exterior planets is* basically correct, except for the fact that mean Sun is used in this model instead of the true Sun. The reason why it works is of course due to the fact that the *Sighra* epicycle would correspond to the apparent orbit of the mean Sun around the Earth, and the *Sighrasamskara* essentially converts the heliocentric longitudes to the geocentric longitudes. This feature is the same as in Ptolemy's model. One difference is in the variable nature of the radius of the epicycle in the Indian model.

Planet	*Aryabhatiya*	Modern value
Mercury	0.361 - 0.387	0.387
Venus	0.712 - 0.737	0.723
Mars	0.637 - 0.662	0.656
Jupiter	0.187 - 0.200	0.192
Saturn	0.114 - 0.162	0.105

Table 3: Comparison of *r/R* (variable) in *Aryabhatiya* for *Sighrasamskara* with modern values of the ratio of the mean values of Earth-Sun and planet-Sun distances for exterior planets and the inverse ratio for interior planets.

For the interior planets, the *sighroccha* is essentially the mean heliocentric planet (θ_M), so that $\theta_{Si} = \theta_M$. θ_{MS} is the *Mandasphuta-graha,* which is the mean Sun corrected for the equation of centre of the planet. Then, Eq.(46) is the same as, Eq.(31) except that θ_{MS} is not the true Sun θ_S, and θ_{Si} is not the true heliocentric planet θ_h, but the mean heliocentric planet. Hence the formulation of the equation of centre for the interior planets is not correct in the traditional Indian model, a feature it shares with Ptolemy's model.

As we have already mentioned there is a clear separation between the exterior and the interior planets in the Indian model, unlike Ptolemy's

model where Mercury alone is treated differently. Also the radii of the epicycles vary in the Indian model. There is another important difference in the actual application of the two equations (*Mandaphala* and *Sighraphala*) to the mean longitude of a planet to find its true longitude. For instance, the following procedure is prescribed to obtain the true geocentric longitudes of the exterior planets in *Aryabhatiya*. First, half the *Mandaphala* is applied to the mean longitude. Second, half the *Sighraphala* obtained using this corrected longitude is applied to it. Third, the *Mandaphala* obtained from this twice corrected longitude is applied to the mean longitude of the planet. This is known as *Sphutamadhya*. Finally, *Sighraphala* calculated using the *Sphutamadhya* is applied to it to obtain the true geocentric longitude. The motivation for the four steps instead of two is not clear. It is possible that the true longitudes computed using the prescribed parameters of the planets and the above procedure were in closer agreement with the observed values, than the ones computed using two steps. It again shows that a rigid geometrical model was not insisted upon.

(d) Planetary latitudes

In the Indian texts, the geocentric latitude, β_E is found using the formula,

$$\beta_E = \frac{i R \sin(\lambda_P - \lambda_N)}{Sighrakarna} = \frac{i R \sin(\lambda_P - \lambda_N)}{EP}, \tag{47}$$

where i is the 'inclination' of the planetary orbit, λ_N is the longitude of the node and λ_P is the *Mandasphutagraha*, which is the true heliocentric planet for the exterior planets [8,9]. For the interior planets, λ_P is the *sighroccha* to which the *Mandaphala* is added. In this case, *sighroccha* is the mean heliocentric planet alright. But the *Mandaphala* is calculated using the mean Sun as the *Madhyamagraha*, before Nilakantha. So, λ_P is not the true heliocentric planet. For the exterior planets, Eq.(47) is the same as Eq.(36) for the geocentric latitude in Kepler's model, if we identify i with i_h, the inclination of the heliocentric orbit with the ecliptic, as SP = the planet-Sun distance can be identified with R (*Trijya*) in the Indian model. For the interior planets, SP in Eq.(36) should be identified with the radius of the *Sighra* epicycle r. However it is R, the radius of the deferent circle which appears in Eq.(47). Hence i cannot be identified with i_h.

In Table 4, we compare the values of i given in *Aryabhatiya* with the inclination of the heliocentric orbits, i_h, as per modern astronomy. For the interior planets, we also give the extrapolated values of $i_e = iR/r$.

Planet	i (*Aryabhatiya*)	i_e (Extrapolated) (*Aryabhatiya*)	i_h Modern
Mercury	2°	5° 10'	7°
Venus	2°	2° 46'	3° 24'
Mars	1° 30'		1° 51'
Jupiter	1°		1° 18'
Saturn	2°		2° 29'

Table 4: Inclinations of the planetary orbits.

For the exterior planets, the values of the inclination i are in reasonable agreement with the inclinations of the heliocentric orbits, as per modern astronomy. The model is essentially correct, and equivalent to a model in which the line of nodes passes through the mean Sun, though it is not explicitly stated to be so. This is in contrast to Ptolemy's model, where it is explicitly stated that the line of nodes passes through the centre of the Earth which is clearly incorrect.

For the interior planets, the equation of centre is applied to the *sighroccha* (mean heliocentric planet) . This is noteworthy, considering the fact that this equation of centre was wrongly applied to the mean Sun, in the calculation of the longitude. However, λ_P is still not the true heliocentric longitude, as noted earlier. The geometrical interpretation of Eq.(47) is not clear, though it is not far off the mark computationally, if we identify i with $i_h r / R$, where i_h is the inclination of the heliocentric orbit with the ecliptic. We have already noted that the geometrical model of Ptolemy for the latitudes of the interior planets is too complicated and has no resemblance whatsoever with Kepler's model.

6. Planetary model of Copernicus

In his *De Revolutionibus* (1543 AD) Copernicus propounded the heliocentric model in which the Sun is at the centre of the planetary orbits, including that of the Earth [10]. The annual motion of the Sun is replaced by that of the Earth. In fact, the Sun is at the centre of the universe, according to him. The model is broadly correct to scale, as the values of the relative mean distances of the planets and the Earth from the Sun are reasonably close to the modern values. One would have expected a simple scheme to calculate the longitudes and latitudes of planets in the model of Copernicus. However, this was not to be. Though the overall framework was heliocentric, Copernicus fashioned his model too closely after Ptolemy. According to the eminent historians of mathematical astronomy,

N.M. Sverdlow and O. Neugebauer,

> ...at no point, however did he [Copernicus] question the soundness of Ptolemy's models for representing the apparent motions of the planets, and so at no time did he carry out the sort of analysis that Ptolemy had, and that Kepler did later, to determine what really constituted an approximate model for the planets[11].

Apart from Ptolemy, there was another decisive influence on Copernicus in his formulation of the planetary model. Astronomers were not comfortable with the motion of the *equant* introduced by Ptolemy, as it violated the principle that all motion should be composed of uniform circular motions. The 'Maragha School' of Islamic astronomers which flourished in Maragha in north-east Iran during 13th and 14th centuries was active in developing planetary models consistent with the principle of uniform circular motion. Muayyad ad-Din al-Urdi (d. 1266), Nasir ad-Din al -Tusi (d. 1274), ash-Shirazi (d 1311) and Ibn ash-Shatir (d. 1375) were some of the celebrated names associated with this school. The method of the Maragha planetary models was to break up the equant motion in Ptolemy's models into two or more components of uniform circular motion, physically the uniform rotation of spheres, that together control the direction and distance of the centre of the epicycle, so that it comes to lie in nearly the same position it would have in Ptolemy's model, and always moves uniformly with respect to the equant [10]. Copernicus's planetary models in his *Commentariolus* (~1514 AD), a short work in which he had outlined the heliocentric theory earlier, are identical to those of Ibn ash-Shatir. In *De Revolutionibus*, the models are somewhat altered, but still based upon the Maragha school. We will not give the details of the Maragha model here.

The computation of the true geocentric longitude of an exterior planet in Ptolemy- Maragha model and in Copernicus's model are compared in Fig.12. O and S_0 refer to the Earth and mean Sun respectively. In the geocentric Ptolemy-Maragha model, C refers to the mean planet corrected for the equation of centre. C is the centre of the epicycle on which the planet P moves. CP is in the same direction as OS_0. The model of Copernicus is essentially a heliocentric representation of this. P is located with respect to the mean Sun S_0, by applying the equation of centre to the mean heliocentric planet. The Earth O is moving in a circle which has the same dimension as the epicycle. It is obvious that the geocentric longitude of the planet is the same in both the models, and is essentially correct, except that the mean Sun has to be replaced by the true Sun.

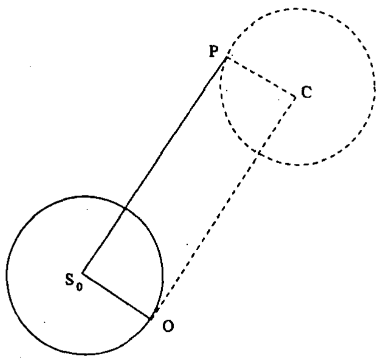

Fig. 12: Location of an exterior planet P in Ptolemy-Maragha model (dotted line) and Copernicus model (solid line)

The location of an interior planet in the two models are compared in Fig.13. In the geocentric picture, *S* refers to the Sun to which the equation of centre of the planet has been applied (wrongly, as described earlier). The interior planet *P* is moving in an epicycle around *S*. In the heliocentric representation of Copernicus, planet *P* is located first and *O'* refers to the Earth to which the equation of centre of the planet has been applied. Hence the formulation of the equation of centre for an interior planet is

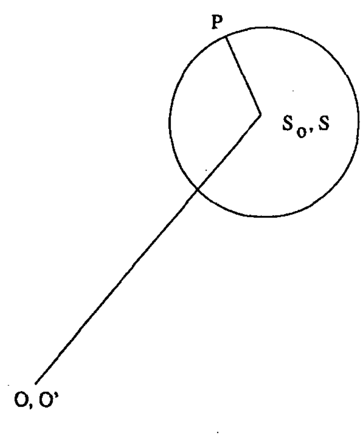

Fig. 13: Location of an interior planet in heliocentric and geocentric pictures.

wrong in the model of Copernicus, as he makes a heliocentric transformation of Ptolemy's model mechanically, ignoring the fact that thereby, the equation of centre has nothing to do with the planet but depends upon the motion of the Earth!

In the words of Sverdlow and Neugebauer [12]:

> What finally is most notable about Copernicus's treatment of interior planets is that he adapts his models and parameters to Ptolemy's even if it might appear peculiar to do so. The equation of centre in Ptolemy's models depends upon the distance of the centre of the epicycle from the planet's apsidal line. This appears odd, at least to us for we know better, but it evidently did not appear odd to Copernicus, who assumed that Ptolemy's models produced essentially correct apparent motions that he wished to reproduce in his own.

It was in the formulation of a theory of latitudes of planets that Copernicus faced the greatest difficulty. The problem was that Copernicus was attempting to reproduce the latitudes of Ptolemy's models and tables in *Almagest*, but his own models would not do it. We have already mentioned that in *Almagest*, for an exterior planet, the inclination of the epicycle to the eccentric was a variable. In the Copernican theory the epicycle represents the orbit of the Earth along the ecliptic, so the orbital plane of the planet, which corresponds to plane of the eccentric in Ptolemy's model has a variable inclination with the ecliptic. So, the orbital plane oscillates in accordance with the motion of the Earth, though the line of nodes passes through the mean Sun. This, of course, has no relation with reality. As we have noted earlier, Ptolemy refined his latitude theory in his *Planetary Hypothesis*, where he eliminated the oscillations of the epicyclic planes altogether and gave them fixed inclinations. For the exterior planets, the epicyclic plane was parallel to the ecliptic. If Copernicus had followed this later text of Ptolemy, he could have avoided the undersirable oscillations of the orbital planes.

The latitude theory of interior planets is even more troublesome, as Ptolemy's model is very complicated and totally off the mark. Again Copernicus tries to adhere to Ptolemy's model, though in a heliocentric framework and it is not surprising that his model is also wrong. In fact, Copernicus's model for latitudes of interior planets is not equivalent to Ptolemy's model for all positions of the Earth. Even then he reproduces the tables of latitudes computed from Ptolemy's model, though they are not fully compatible with computations based on his own model.

Sverdlow and Neugebauer comment on the latitude theory of Copernicus thus [13]:

> "Copernicus, ignorant of his own riches, took it upon himself for the most part to represent Ptolemy, not nature, to which he had nevertheless come the closest of all". In this famous, and just, assessment of Copernicus, Kepler was referring to the latitude theory of Book V, specifically to the "librations" of the inclinations of the planes of the eccentrics, not in accordance with the motion of the planet, but (*quod monstri simile est*) by the unrelated motion of the earth. This improbable connection between the inclinations of the orbital planes and the motion of the earth was the result of Copernicus's attempt to duplicate the apparent latitudes of Ptolemy's models in which the inclinations of the epicyclic planes were variable. In a way this is nothing new since Copernicus was also forced to make the equation of centre of the interior planets depend upon the motion of the earth rather than the planet. But the image of the orbital planes of all the planets dutifully inclining one way and another in synchronization with the motion of the earth was to Kepler an incongruous combining of unrelated motions that he found incredible even before he was able to show, from Tycho's observations, that it was false.

De Revolutionibus was hailed as opening a new era, by a small group of astronomers in Europe, but the heliocentric system did not find acceptance by and large till the work of Kepler. Prominent among the opponents of the idea of Earth's annual motion (in fact its rotation also) was Tycho Brahe who was a meticulous observer. In the model of Tycho Brahe (1583 AD), the Earth is the centre of the universe and the centre of the orbit of the Moon and the Sun. The Sun is the centre of the orbits of the five planets. This system is in reality absolutely identical with the system of Copernicus, except that it is geocentric, and all computations of the places of planets are nearly identical. It shares the same defects as the Copernican model, in that the formulations of equation of centre for interior planets and the latitude theory are wrong.

Historians of astronomy have wondered why the Tychonic system which might serve as a stepping stone from the Ptolemaic to the Copernican system was not proposed before the latter. A system which lead to the Tychonic geometrical picture of planetary motion, was indeed proposed earlier, in a different setting and framework, without many of the defects in it or the Copernican system – defects which were in fact inherited from their adherence to the Ptolemaic model. That was the planetary model of Nilakantha proposed in his *Tantrasangraha* (1500 AD) and other works.

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Nilakantha's Revision of the Traditional Indian Planetary Model

K.Ramasubramanian

Introduction

There has been continuous progress in the Indian astronomical tradition, from the time of *Vedanga-Jyotisha* by Lagadha (prior to 14th cent. BC) to Samanta Chandrasekhara (19th cent. AD). Particularly, the advances made by the Kerala school of astronomers and mathematicians between 14th-18th centuries are indeed remarkable.

Nilakantha Somayaji (1444-1545 AD), a versatile scholar of this school, has made significant contributions both in astronomy and mathematics. His erudite commentary on *Aryabhatiya* is an outstanding work and stands as a testimony for his mastery over several *sastras*. In this work, Nilakantha elaborately discusses all the minute details essential for a clear and thorough understanding of the principles involved in the computation of planetary positions, eclipses, etc. Another important work of Nilakantha is *Tantrasangraha*, which is a full-fledged text on Indian astronomy consisting of 432 verses. It is in this work that Nilakantha gives the details of the major revision of the traditional Indian planetary model as carried out by him.

This paper discusses the planetary model introduced by Nilakantha in *Tantrasangraha*[1] and further elaborated in his *Aryabhatiyabhashya*[2]. We shall highlight two of his most significant contributions, namely (i) the unified formulation for the computation of planetary latitudes and (ii) the proper application of *equation of centre* in the case of the interior planets. These are indeed major landmarks in the history of astronomy.

[1] Critically ed.with two commentaries by K.V. Sarma, Visveshvaranand Inst. Panjab Univ. Horshiarpur, 1977.

[2] Published in 3 volumes, Trivandrum Sanskrit Series, 101, 110, 185 (Trivandrum, 1930, 1931, 1957).

Computation of planetary latitudes

The plane in which the Sun appears to move around the Earth is referred to as the ecliptic. The orbital planes of all the planets are slightly inclined to this plane. For instance, Mercury's orbital plane is inclined to this plane by of about 7°. The inclination of the orbital planes of other planets ranges between 1 - 3.5°. The deflection of the planet (in angular measure) from the ecliptic at any instant, is referred to as the latitude of the planet. Since the planets are continously moving in their orbits, the latitude is a continuously varying quantity. The maximum latitude attained by a planet is equal to the angle of inclination of the orbital plane to the ecliptic. In Fig. 1, we schematically represent the planetary orbit and the ecliptic.

The orbit of the planet intersects the ecliptic at two points. These two points are referred to as the nodes of the planetary orbit. In Fig. 1, *P* refers to the position of the planet and *N* refers to the ascending node of the planet. The point *Q* represents the foot of perpendicular drawn from *P* to the ecliptic. $PQ(\beta)$ is referred to as the latitude of the planet. Let i be the angle of inclination of the planetary orbit to the ecliptic.

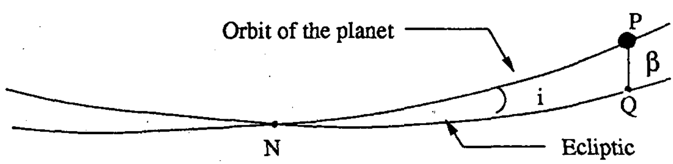

Fig. 1: Orbit of a planet inclined at an angle i with respect to the ecliptic.

The relationship between β, i and the arc *PN* is:

$$\sin\beta = \sin i \sin PN \qquad (1)$$

The above relation is an exact result obtained from the properties satisfied by the spherical triangle *NPQ*. The formula given by Indian astronomers for the computation of planetary latitudes is:

$$\sin\beta = i \sin NQ, \qquad (2)$$

where *NQ* represents the difference in the longitude of the planet and its node. For small inclinations, $\sin i \approx i$ and $PN \approx NQ$. Hence, to a good degree of approximation, Eq.(1) reduces to Eq.(2). The latitude obtained

from Eq.(2) is the heliocentric latitude. The procedure used by the Indian astronomers to get the geocentric latitude from the heliocentric latitude will be discussed later in this article.

The modern values of the mean orbital inclinations of the planets and the ones given in *Aryabhatiya* are listed in Table 1. For all the planets, except Mercury, the orbital inclination is very small, less than 3.5°.

	Mercury	Venus	Mars	Jupiter	Saturn
Modern	7.0	3.3	1.8	1.3	2.4
Aryabhatiya	2.0	2.0	1.5	1.0	2.0

Table 1: Mean orbital inclinations of the planets.

From Table 1, it may be observed that, for the exterior planets Mars, Jupiter and Saturn, the values given in *Aryabhatiya* are not very different from the modern values, whereas there is a considerable difference in the case of the interior planets Mercury and Venus[3].

The concept of Sighroccha

In Indian astronomy, computation of the geocentric longitude of a planet, referred to as the *sphutagraha,* is carried out in two steps:

i. computation of the true heliocentric longitude (*mandasphutagraha*)
ii. transformation of this true heliocentric longitude into the geocentric longitude.

In the second step, which is termed *sighrasamskara,* the Sun plays a pivotal role. While the Sun is referred to as the *sighroccha* in the case of exterior planets, the mean heliocentric planet has been traditionally referred to as the *sighroccha,* in the case of interior planets.

The sidereal period of a planet is the time interval between two successive passages of the planet over a particular reference point in the Zodiac. In Indian astronomy, this reference point is taken to be the beginning point of the *Aswini nakshatra.* In the case of exterior planets, since the orbit of the planet circumscribes the Earth, the mean geocentric sidereal period will be the same as the heliocentric sidereal period. But

[3] For a more detailed discussion, refer to the article by M.S. Sriram on *Planetary Models in Indian* and *Greek Astronomical Traditions,* in these proceedings.

in the case of interior planets, this is not so because their orbits do not circumscribe the Earth. Their motion around the Earth is only due to the motion of the Sun around the Earth. Hence the mean geocentric sidereal period of Mercury or Venus is the same as that of the Sun, which is one year. However, the heliocentric sidereal periods of the interior planets were also known to Indian astronomers, and they have been referred to as the period of the *sighroccha* of the planet. For example, in *Suryasiddhanta* we find the following verse giving the number of revolutions of *sighrocchas* of Mercury and Venus [4] :

> बुधशीघ्रस्य शून्यर्तुखाद्रित्र्यंकनगेन्दव:।
> सितशीघ्रस्य षट्सप्तत्रियमाश्विखभूधरा:।।

> (The revolution numbers) of the *sighrocchas* of Mercury and Venus are 17937060 and 7022376 respectively.

These revolution numbers correspond to the revolutions made by the *sighrocchas* of Mercury and Venus in a very large period of time called a *Mahayuga* consisting of 4320000 years and 1577917828 civil days. The time taken by the planet to make one complete revolution in the background of fixed stars can be obtained by dividing the number of civil days by its revolution number. This is referred to as the sidereal period and in the case of the *sighrocchas* of Mercury and Venus, these turn out to be approximately 87.97 and 224.69 days respectively. Clearly, these periods correspond to the heliocentric sidereal periods of the planets themselves.

Indian astronomers have been referring to these periods as the periods of revolution of the *sighrocchas* of Mercury and Venus. Perhaps this is because their angular velocity is greater than that of the Sun, which is involved in the *sighrasamskara* of the planet. In the case of the exterior planets, since the *gati* or *bhukti* (rate of angular motion) of the Sun is greater than that of the planet, the Sun is referred to as the *sighroccha* of the planet and the heliocentric planet is referred to as the *mandasphuta*. But in the case of the interior planets, since the planet moves faster than the Sun, the heliocentric planet is referred to as the *sighroccha* of the planet and the mean Sun is referred to as the mean planet.

[4] *Suryasiddhanta,* 1. 31-32.

Different rules for the computation of latitudes

The Indian astronomers, right from the time of Aryabhata (5th cent. AD), were using two seemingly different rules for the computation of planetary latitudes, one for the interior planets, namely Mercury and Venus, and one for the exterior ones. The formula used for the exterior planets was

$$\sin \beta = i \sin (\lambda_m - \lambda_n), \tag{3}$$

where λ_n refers to the longitude of the node. λ_m refers to the *mandasphuta,* or the longitude of the true heliocentric planet, which is the mean heliocentric longitude corrected by the equation of centre. The formula used for an interior planet was

$$\sin \beta = i \sin (\lambda_s - \lambda_n), \tag{4}$$

where λ_s refers to the longitude of the *sighroccha* of the planet corrected by the *mandaphala* and λ_n refers to the longitude of the node. As mentioned in the previous section, since the mean heliocentric planet was referred to as the *sighroccha* in the case of interior planets, the above formula would have yielded a reasonably correct value for the latitude of the planet.

This dual formulation for the computation of latitudes of planets continued to be employed in Indian astronomy, till Nilakantha revised the traditional Indian planetary model and established a unified formulation for the computation of the latitudes of all the five planets. *Aryabhatiya* lays down the procedure for the computation of latitudes as follows[5]:

अपमण्डलस्य चन्द्रः पाताद् यात्युत्तरेण दक्षिणतः।
कुजगुरुकोणाश्चैवं शीघ्रोच्चेनापि बुधशुक्रौ।।

From the node of the planet on the ecliptic, the moon moves towards the north or south (of the ecliptic). The planets Mars, Jupiter and Saturn have a similar motion. Mercury and Venus also have a similar motion due to their *sighrocchas*.

Here we find Aryabhata making a clear distinction between the interior and exterior planets. While he associates the latitudinal motion to the planets themselves in the case of extetior planets, he associates the same with the *sighroccha,* in the case of interior planets. The successors of Aryabhata like Manjulacharya and Lallacharya essentially followed

[5] *Aryabhatiya,* Golapada.3.

the same prescription. For instance, Manjulacharya prescribes the rule for the computation of latitudes as follows[6]:

> मन्दस्फुटात् स्वपातोनाद् ग्रहाच्छीघ्रात् ज्ञशुक्रयो:।
>
> (The latitude of the planet has to be determined) by subtracting the node from the *mandasphuta* (in the case of exterior planets) and from the *sighroccha* of Mercury and Venus.

Lallacharya in his *Sishyadhivriddhidatantram* states[7]:

. . . समलिप्तिकाद् विपातात्। त्रिदशगुरुमहीजसूर्यजातानाम् भृगुतनयेन्दुजयोस्तथैव शीघ्रात्।। निजमध्यमबाणसंगुणा भुजजीवा कुखगान्तरोद्धृता। स शरो भवति . . .।।

> The sine of the difference in the longitudes of the Mars, Jupiter, Saturn, and their nodes, and similarly the sine of the difference in the longitude of the *sighroccha* of Mercury and Venus and their nodes, multiplied by their orbital inclination, and divided by the distance of separation between the planet and the Earth, gives the latitude of the planet.

Several Indian astronomers were acutely aware that they were using a dual formulation for the computation of the latitudes namely, Eqns.(3) and (4). For instance, Bhaskaracarya (12th cent.) who has adopted this dual formulation, makes the following observation in his *Vasanabhashya*, (a pointed and brief commentary on his own monumental work *Siddhantasiromani*, explaining the rationale behind the procedures prescribed in the text) [8]:

ननु ज्ञशुक्रयो: शीघ्रोच्चपातयुतिं केन्द्रं कृत्वा यो विक्षेप आनीत: स शीघ्रोच्चस्थाने एव भवितुमर्हति; न ग्रहस्थाने; यतो ग्रहोऽन्यत्र वर्तते। अत इदमनुपपन्नमिव प्रतिभाति; तथा च ब्रह्मसिद्धान्तभाष्ये ज्ञशुक्रयो: शीघ्रोज्चस्थाने यावान् विक्षेप: तावानेव यत्र तत्रस्थस्यापि ग्रहस्य भवति; अत्रोपलब्धिरेव वासना नान्यत् कारणं वक्तुं शक्यत इति चतुर्वेदेनाप्य-ध्यवसायोऽत्र कृत:।

> The deflection that is obtained by using the *sighroccha* and the node, must be the latitude at the location of *sighroccha* and not at the location of the planet, as the planet is somewhere else. Therefore this (procedure used for the computation of the latitudes for the interior planets) seems to be inappropriate. However, even Chaturvedacharya (Prithudakaswamin) has concluded as follows in his commentary (*Vasanabhashya*) on *Brahmasphutasiddhanta*: The deflection at the location of the *sighroccha* of the Mercury and Venus, corresponds to the latitude of the planet itself, wherever

[6] *Laghumanasa* 3.6.

[7] *Sishyadhivriddhidam*, Grahaganita, Grahayuti. 9.

[8] *Vasanabhashya*, Golabandhadhikara, Ver.23-24.

it be; here *upalabdhi*, the fact that the calculated results agree with observations, is the only justification [for this procedure] as we are unable to give any other reason.

Most Indian astronomers seem to have reconciled themselves to the dictum of Prithudakaswamin (860 AD) that we have to live with this seemingly different procedures for computing the latitude of an interior planet, as they led to results which are in accordance with the observed values.

This agreement between the computed values and the observed values is mainly because the Indian astronomers were using the *sighroccha* for the computation of latitudes of the interior planets, which (as we saw) refers to the mean heliocentric planet. Though in the traditional Indian planetary model, the equation of centre was wrongly applied to the mean Sun, the latitude of the planet was calculated from the *sighroccha*. Hence, the latitude values did not go far off the mark.

This may be contrasted with the computation of latitudes in the Greek tradition. In the Greek planetary model proposed by Ptolemy, the line of nodes of all the planets were assumed to pass through the centre of the Earth. Hence, the latitudinal calculations were completely off the mark particularly in the case of interior planets. Not only were they off the mark, the model was extremely complicated too[9].

Nilakantha's analysis of the problem

Like Bhaskaracharya, the famous Kerala astronomer Nilakantha Somayaji also discusses the dual formulation used in the computation of latitudes of the planets, in his *Aryabhatiyabhashya*. However, he does not remain satisfied with the the dictum of Prithudakaswamin that we have to live with two different methods for the computation of the latitudes. Instead, he arrives at a unified formulation for the computation of latitudes of all the planets as described below.

Nilakantha argues that the latitude of a planet has to be a measure of the deflection of the planet from the ecliptic and not of some other entity namely, *sighroccha;* therefore he proposes that what was taken as the *sighroccha* in the case of an interior planet in the traditional model, should

[9] For more details regarding the Greek planetary model, refer to the article by M.S. Sriram in these proceedings.

be identified with the planet itself; he further argues that the mean Sun should be taken as the *sighroccha* of all the planets. He states[10]:

बुधशुक्रावपि स्वस्वपातात् प्रभृत्यपमण्डलादुत्तरतो दक्षिणतश्च यातः. . . . । बुधस्य द्वाविंशत्यैवाहोरात्रैर्वर्धमानस्य विक्षेपस्य महत्त्वनिवृत्तिः स्यात्; पुनः ह्रासवशात् तावद्भिरेव दिनैः शून्यतापि स्यात्। एवमपक्रममण्डलादेकपार्श्वे एव गमनं चतुश्चत्वारिंशद्दिनान्येव; पुनरितरपार्श्वे तावन्त्येव दिनानि गच्छति। एवमष्टाशीत्यैव दिनैर्विक्षेपस्यैकः पर्यायः परिसमाप्तः स्यात्; यतोऽष्टाशीत्यैव दिनैः शीघ्रभगणपरिपूर्तिः।

शीघ्रवशाच्च विक्षेप उक्तः; कथमेतद्युज्यते। ननु स्वबिंबस्य विक्षेपः स्वभ्रमणवशादेव भवितुमर्हति। न पुनरन्यभ्रमणवशादिति। सत्यम्। न पुनरन्यस्य भ्रमणवशादन्यस्य विक्षेपः उपपद्यते। तस्माद्बुधोऽष्टाशीत्यैव दिनैः स्वभ्रमणवृत्तं पूरयति। ...भगोलपरिभ्रमणं तस्याप्येकेनैवाब्देन। . . . . शुक्रोऽपि दिनानां पञ्चविंशत्यधिकशतद्वयेन स्वभ्रमणवृत्तं पूरयति। . . . तयोरपि वस्तुतः आदित्यमध्यम एव शीघ्रोच्चम्।

Even Mercury and Venus move towards the North and South of the ecliptic from their own nodes. ...The latitudinal deflection of Mercury (from the ecliptic) increases continuously for about 22 days, till it gets stalled; again in the same number of days, the latitudinal deflection becomes zero due to a continuous decrease. Thus the motion (of Mercury) on one side of the ecliptic is only for 44 days. It moves on the other side of the ecliptic for the same number of days only. Thus one round of latitudinal deflection would have got completed in just 88 days as the *sighroccha* (of Mercury) completes its revolution in 88 days.

The deflection is stated to be due to the *sighroccha*. How is it appropriate? The deflection of an object can take place only due to its own motion and not the motion of something else. True it is; that the deflection of some object cannot take place due to the motion of something else. Therefore Mercury completes its own orbit in 88 days....The period in which it (Mercury) also completes one full revolution around the *bhagola* (celestial sphere) is one year only....In the same way Venus also completes its orbit in 225 days only....In fact the mean Sun is the *sighroccha* for them (Mecury and Venus) also.

In his treatise *Tantrasangraha*, Nilakantha makes it very clear that what were referred to as the *sighrocchas* of Mercury and Venus by the ancients are indeed the mean planets themselves. While giving the number of revolution of planets in a *Mahayuga* he states[11]:

खाश्विदेवेषुसप्ताद्रिशराश्चेन्दोः कुजस्य तु।
वेदांगहिरसांकाश्विकरा ज्ञस्य स्वपर्ययाः।।
नागवेदनभःसप्तरामांकस्वरभूमयः।

[10] *Aryabhatiyabhashya*. Golapada, verse.3
[11] *Tantrasangraha*, I.16-18.

व्योमाष्टरूपवेदांगपावकाश्च बृहस्पतेः।।
अष्टांगदस्रनेत्राश्विखाद्रयो भृगुपर्यया:।

The number of revolutions of the Moon is 57753320; and that of Mars is 2296864. Mercury's own revolutions is 17937048; that of Jupiter is 364180. The revolutions of Venus is 7022268.

Here the use of the term '*svaparyaya*' (its own revolution), while giving the number of revolutions of *Budha* (Mercury) and '*Bhriguparyaya*' while giving the revolutions of *Bhrigu* (Venus), are to be noted. As mentioned earlier, these revolution numbers are normally associated with the *sighrocchas* of Mercury and Venus. This important change introduced by Nilakantha is highlighted by Sankara Varier in his commentaries, *Yuktidipika* and *Laghuvivritti*. For instance, in *Laghuvivritti*, Sankara Varier observes:

अत्र स्वशब्देन पर्यायाणां भास्कराचार्याद्यभिमतं स्वशीघ्रोच्चसम्बन्धित्वं बुधस्य निरस्तम्।

By the word '*sva*' the connection of *Budha* (revolution number of Mercury) with its *sighroccha* as given by Bhaskaracarya I and others has been discarded.

In *Yuktidipika* he states[12]:

बुधभार्गवयोर्वृत्तं शैघ्र्यनीचोच्चवृत्तताम्।
याति तस्मिन् स्वगत्यैव चरतो बुधशुक्रयोः।।
युगोत्थाः पर्ययास्त्वेते तत्स्वशब्देन दर्शितम्।

The orbits of Mercury and Venus become the *sighranichocchavritta* (employed in *sighra* correction). The number of revolutions given (17937048 and 7022268) correspond to the revolution made by the planets (in a *Mahayuga*) moving in their own orbits. This is indicated by the use of the word *sva*.

In a short anonymous tract, *Vikshepagolavasana*, it is clearly stated that Nilakantha was the first astronomer to propose that what were taken earlier as the *sighrocchas* of *Budha* and *Sukra* were in fact the planets themselves.

पूर्वाचार्यैस्तु मान्दे अपि खलु परिधी भानुकक्ष्याकलाभि-
र्मत्वोक्ते तेन मान्देऽपि च दिनकरमध्यं स्वमध्यं प्रदिष्टम्।

[12] *Yuktidipika* I. 69-70.

मन्दोच्चोनार्कमध्यात् उदितमृदुफलं क्षेपनीतौ चलोच्चे
कुर्वन्त्येतन्न युक्तं तदकरणमतो मानसे युक्तिमत् स्यात्।।

कुर्वन्त्यस्मिन् हि पक्षे तदिदमनुचितं भिन्नजातित्वहेतोः
तस्मात् गार्ग्येण मान्दे शशिसुतसितयोर्मध्यमं स्वीयमध्यम्।
प्रोक्तं मान्दञ्च वृत्तं प्रमितमिह तयोः स्वीयकक्ष्या कलाभिः
शैघ्रे स्वान्मध्यवृत्ताद्दिनकरवलयस्याधिकत्वेन युक्त्या।।

By taking the measure of the *manda* deferent circle (of Mercury and Venus) to be equal to that of Sun, it has been stated by the earlier *Acharyas* (prior to Nilakantha) that the mean Sun is the mean planet (in the case of Mercury and Venus). They (*Acharyas*) apply the *mandaphala*, obtained by subtracting the *mandoccha* from the mean Sun, to the *sighroccha* (of Mercury and Venus) while determining their latitudes. This (procedure) is not appropriate. Hence, in the *Laghumanasa* the non-adoption of this procedure seems to be appropriate.

Such a procedure is inappropriate because of the dual formulation (in the computation of latitudes); Hence it has been stated by Nilakantha that (even) in *manda samskara* the mean positions of Mercury and Venus are found from their own *madhyamas* (and not that of Sun). Since the orbit of the Sun involved in the *sighra* correction is larger than their own *manda* deferent circle, the *manda* deferent circle has been measured in terms of the dimensions of their own orbits.

Parameswara, the *Paramaguru* of Nilakantha seems to have indicated in his work that what was referred to as *sighroccha* was indeed the mean planet in the case of Mercury and Venus. For instance, he states in his work *Drigganita* [13]:

शीघ्रोच्चं रविमध्यं कुजस्य, तद्वद् गुरोः शनेश्चापि।
निजमध्यं शीघ्रोच्चं बुधसितयोर्भानुमध्यमं मध्यम्।।

The mean Sun is the *sighroccha* for Mars, Jupiter and Saturn. The mean Sun is the mean planet and the actual mean planet is the *sighroccha* in the case of Mercury and Venus.

[13] *Drigganitam* III.3

Nilakantha's planetary model

(a) Exterior planets

Nilakantha's planetary model for exterior planets is not different from the traditional Indian planetary model. In the traditional model, the *mandaphala* or the equation of centre is applied to the *madhyamagraha* or the mean planet, which is also the mean heliocentric longitude of the planet, to obtain the *mandasphutagraha* or the true heliocentric longitude of the planet. By performing *sighrasamskara* on the *mandasphuta-graha,* one obtains the *sphutagraha* or the geocentric longitude of the planet. Nilakantha in his *Tantrasangraha*, describes the procedure for the computation of geocentric longitudes of the exterior planets with the following verses [14]:

> दोःकोटिज्याष्टमांशौ स्वखाब्ध्यंशोनौ शनेः फले।
> दोर्ज्या त्रिज्याप्तसप्तैक्यं गुणो मान्दे कुजेड्ययोः॥
> नवाग्नयो द्व्यशीतिश्च हारौ दोःकोटिजीवयोः।
> पृथक्स्थे मध्यमे कार्यं दोःफलस्य धनुर्दलम्॥
> रविमध्यं विशोध्यास्मात् पृथक्स्थाद् बाहुकोटिके।
> आनीय बाहुजीवायास्त्रिज्याप्तं गुरुमन्दयोः॥
> षोडशभ्यो नवभ्यश्च कुजस्यापि स्वदोर्गुणात्।
> त्रिज्याप्तं द्विगुणं शोध्यं त्रीषुभ्यः शिष्यते गुणः॥
> अशीतिरेव तेषां हि हारस्ताभ्यां फले उभे।
> आनीय पूर्ववत्कर्णं सकृत् कृत्वाथ दोःफलम्॥
> त्रिज्याघ्नं कर्णभक्तं यत् तद्धनुर्दलमेव च।
> मध्यमे कृतमान्दे तु संस्कृत्यातो विशोधयेत्॥
> मन्दोच्चं तत्फलं कृत्स्नं कुर्यात् केवलमध्यमे।
> तस्मात् पृथक्कृताच्छैघ्रं प्राग्वदानीय चापितम्॥
> कृतमान्दे तु कर्तव्यं सकलं स्यात् स्फुटः स च॥

One-eighth of the *dorjya* and *kotijya* (sine and cosine of the *mandakendra*) diminished by one-fortieth of the same, form the *dohphala* and *kotiphala* in the case of Saturn. The *dorjya* divided by *trijya* and added to 7, forms the *guna* (multplier/numerator) for Mars and Jupiter. 39 and 82 are the *hara* (divisor/denominator) for Mars and Jupiter respectively. Half of the

[14] *Tantrasangraha* II 61-67.

arc of the *dohphala* has to be applied to the mean longitude of the planet (l_0) to get the first corrected longitude (l_1).

Subtracting the longitude of the Sun from this (l_1), the *dorjya* and *kotijya* are obtained. Dividing the *dorjya* by *trijya* and subtracting from 16 and 9, we get the *guna* for Jupiter and Mars respectively. The same (*dorjya*) multiplied by 2 and subtracted from 53 forms the *guna* for Mars.

80 is the *hara* for all of them (in the *sighra samskara*). From them (*guna* and *hara* of all the three planets) after obtaining the *dohphala* and *kotiphala*, and the *sakritkarna* (once calculated hypotenuse), half of the *dohphala* multiplied by *trijya* and divided by *karna* is applied to the first corrected longitude (l_1). (The longitude thus obtained is, say l_2). From this (l_2), let the *mandoccha* be subtracted and the full *mandaphala* be obtained; let that be applied to original mean planet (l_0) (to get, say l_3). From that (l_3) let *sighraphala* be obtained as before, and let this be applied fully to the *manda* corrected planet (l_3). The longitude obtained thus is the *sphuta* (the geocentric longitude of the planet).

In the following, we symbolically represent the above procedure for finding the geocentric longitude of one of the exterior planets, say Mars. In the case of exterior planets, to get the geocentric longitude from the mean heliocentric longitude of the planet, four corrections are applied, namely *half-manda, half-sighra, full-manda* and *full-sighra*. Let l_0 be the mean longitude of the planet at a given time. First, l_1 is obtained by:

$$l_1 = l_0 + \tfrac{1}{2}\sin^{-1}\left(\frac{r}{R}\sin M\right),$$

where M the *mandakendra* = mean longitude of the planet – longitude of its apogee, and the ratio $\frac{r}{R}$ of the *manda* epicycle to the deferent circle is given by

$$\frac{r}{R} = \left(\frac{7 + |\sin M|}{39}\right).$$

Next, l_2 is obtained from l_1 by

$$l_2 = l_1 + \tfrac{1}{2}\sin^{-1}\left(\frac{r\sin S}{karna}\right),$$

where S is the *sighrakendra* (l_1 – longitude of its *sighroccha*), and the ratio of the *sighra* epicycle to the deferent circle is given by

$$\frac{r}{R} = \left(\frac{53 - 2|\sin S|}{80}\right),$$

and

$$karna = \left[(R+r\cos S)^2 + (r\sin S)^2\right]^{1/2}.$$

With l_2 once again, full *manda* correction is done and the full *manda*-corrected longitude l_3, known as the *manda-sphutagraha* (the true heliocentric longitude of the planet) is obtained from the original mean longitude l_0 by

$$l_3 = l_0 + \sin^{-1}\left(\frac{r}{R}\sin M\right),$$

where the *mandakendra* $M = l_2 -$ longitude of its apogee, and

$$\frac{r}{R} = \left(\frac{7+|\sin M|}{39}\right).$$

With the *manda-sphutagraha* l_3 the *sphutagraha*, or the geocentric longitude l_4, is obtained by

$$l_4 = l_3 + \sin^{-1}\left(\frac{r\sin S}{karna}\right),$$

where the *sighrakendra* $S = l_3 -$ longitude of its *sighroccha*,

$$\frac{r}{R} = \left(\frac{53-2|\sin S|}{80}\right),$$

and

$$karna = \left[(R+r\cos S)^2 + (r\sin S)^2\right]^{1/2}.$$

The *sphutagraha* of the other two exterior planets, namely Jupiter and Saturn, are obtained in a similar manner.

Essentially, in verses 61 and 62, the procedure for obtaining *mandaphala* (equation of centre correction) to be applied to the mean planet in the case of Mars, Jupiter and Saturn is given. The formula involved in the procedure has terms containing the ratio of *manda* epicycle to the deferent circle. We notice that Nilakantha has assumed a variable epicycle whose dimensions at the end of even and odd quadrants are specified. He has also assumed the variation to be sinusoidal (see Fig.2).

In verses 63-65, Nilakantha gives the ratio of the *sighra* epicycle to the deferent circle which is used in finding the *sighraphala*. Here again, the dimension of the *sighra* epicyle is assumed to be varying according

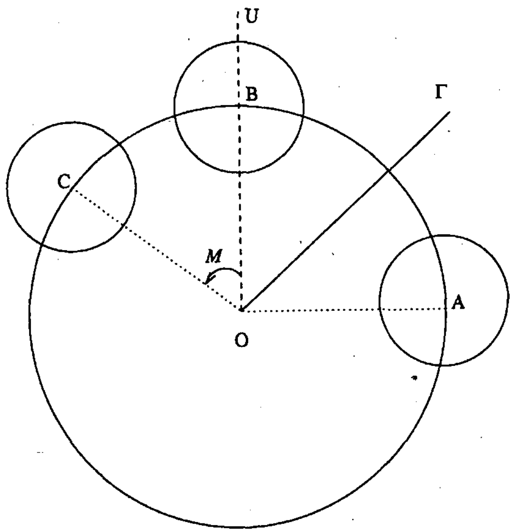

Fig. 2: *Manda-samskara* with variable epicycle model for an exterior planet, say Mars. OU represents the direction of *mandoccha* and M represents *mandakendra*. The radii of the epicycles centered around A, B and C are 8 units, 7 units and 7 + |sin M| units respectively.

to the sine of the *sighrakendra*. The ratios of the *manda* epicycle to the deferent circle and *sighra* epicyle to the deferent circle for the three exterior planets are given in Table.2.

Planet / Ratio	Mars	Jupiter	Saturn
$\frac{r}{R}$ (*Manda*)	$\frac{7+\lvert\sin M\rvert}{39}$	$\frac{7+\lvert\sin M\rvert}{82}$	$\frac{39}{320}$
$\frac{r}{R}$ (*Sighra*)	$\frac{53-2\lvert\sin S\rvert}{80}$	$\frac{16-\lvert\sin S\rvert}{80}$	$\frac{\overset{9}{\cancel{53}}-\cancel{2}\lvert\sin S\rvert}{80}$

Table 2: Ratio of *manda* and *sighra* epicycles to the deferent circle for the exterior planets.

The rationale behind the order of application of *manda* and *sighraphalas* is not apparent to us. Since the *manda* correction takes into account the eccentricity of the planetary orbit, and the *sighra* correction transforms the heliocentric longitude to the geocentric longitude, we would expect the application of *manda* and *sighraphalas* only once. But the text prescribes an additional application of half of the *manda* and *sighraphalas*. This is in line with the procedure outlined in most other Indian astronomical texts which prescribe some such repeated application.

(b) Interior planets

Though Nilakantha's model for the exterior planets is along the standard lines, in the case of *Budha* and *Sukra* it is significantly different. In the second chapter of *Tantrasangraha*, while describing the procedure for the computation of geocentric longitude of the interior planets, Nilakantha, for the first time in the history of astronomy, proposes that the equation of centre (*mandaphala*) should be applied to the mean heliocentric longitude of the planet and not to the mean Sun as was the practice till then. Later in the same chapter, Nilakantha describes the procedure for the computation of geocentric longitudes of the interior planets as follows[15]:

बुधमध्यात् स्वमन्दोच्चं त्यक्त्वा दोःकोटिजीवयोः।।
षडंशाभ्यां फलाभ्यां तु कर्णः कार्योऽविशेषतः।
दोःफलं केवलं स्वर्णं केन्द्रे जूकक्रियादिगे।।
एवं कृतं हि तन्मध्यं स्फुटमध्यं बुधस्य तु।
रविमध्यं ततः शोध्यं दोःकोटिज्ये ततो नयेत्।।
दोर्ज्या द्विघ्ना त्रिभज्याप्ता शोध्यैकत्रिंशतो गुणः।
मन्दकर्णहतस्सोऽपि त्रिज्याप्तः स्यात् स्फुटो गुणः।।
तद्धते बाहुकोटिज्ये खाहिभक्ते फले उभे।
ताभ्यां कर्णं सकृन्नीत्वा त्रिज्याघ्नं दोःफलं हरेत्।।
कर्णेनाप्तस्य यच्चापं कृत्स्नं तद्भानुमध्यमे।
क्रमेण प्रक्षिपेज्जह्यात् केन्द्रे मेषतुलादिगे।।
एवं शीघ्रफलेनैव संस्कृतं रविमध्यमम्।
बुधः स्यात् स स्फुटः शुक्रोऽप्येवमेव स्फुटो भवेत्।।

[15] *Tantrasangraha* II 68-79.

मन्दकेन्द्रभुजा जीवा खजिनांशेन संयुता।
मनवस्तस्य हारः स्यात् तद्धृते बाहुकोटिके।।
स्यातां मन्दफले तस्य दोःफलं च स्वमध्यमे।
कृत्वा/विशेषकर्णं च क्रियतां शीघ्रकर्म च।।
द्विघ्ना दोर्ज्या त्रिभज्याप्ता शोध्यास्यैकोनषष्टितः।
गुणः सोऽपि स्फुटीकार्यो मन्दकर्णेन पूर्ववत्।।
गुणः स मन्दकर्णघ्नस्त्रिज्याप्तस्तस्य च स्फुटः।
अशीत्याप्ते भुजाकोटी तद्घ्ने शीघ्रफले भृगोः।।
दोःफलं त्रिज्यया हत्वा शीघ्रकर्णहृतं भृगोः।
चापितं भास्वतो मध्ये संस्कुर्यात् सः स्फुटः सितः।।

From the *madhyamagraha* of Mercury subtracting the *mandoccha*, *dorjya* and *kotijya* are obtained. From one-sixth of these values, the *karna* is found iteratively. The *dohphala* has to be added to or subtracted from the *madhyamagraha*, depending on whether the *mandakendra* lies within 6 signs of *Tula* or *Mesha*. The value thus obtained is the *mandasphutagraha* of Mercury (say l_1).

Then subtracting the mean Sun (which is the *sighroccha*) from this (l_1), obtain the *dorjya* and *kotijya* (corresponding to *sighrakendra*). The *dorjya* multiplied by 2, divided by *trijya* and subracted from 31 forms the *guna*. This further multiplied by the *karna* and divided by *trijya* forms the *sphutaguna* (true multiplier).

The *dorjya* and *kotijya* multiplied by the *sphutaguna* and divided by 80 forms the *dohphala* and *kotiphala* respectively. From these two (*dohphala* and *kotiphala*), obtaining the *karna* once (not iteratively), divide the product of *trijya* and *dohphala* by this *karna*. The arc of this is fully applied to the mean Sun. It is either added or subtracted depending upon whether the *sighrakendra* lies within 6 signs of *Tula* or *Mesha*. The mean Sun corrected by this *sighraphala* gives the true geocentric longitude of Mercury.

The true geocentric longitude of Venus is also obtained in a similar manner. The 240th part of the R sine of the *mandakendra* added to 14 (forms the *hara*). The *bahujya* and the *kotijya* divided by this divisor form the *dohphala* and *kotiphala* in the *mandasamskara*. Applying the arc of the *dohphala* to the *madhyamagraha*, let the once calculated *karna* and the *sighrasamskara* be carried out.

The *dorjya* (corresponding to *sighrakendra*) multiplied by two, divided by *trijya* and subtracted from 59, forms the *guna*. This multiplied by the *mandakarna* and divided by *trijya* forms the *sphutaguna*. The *dorjya* and *kotijya* multiplied by the *sphutaguna* and divided by 80 are the *dohphala* and *kotiphala*. The arc of the *dohphala* multiplied by *trijya* and divided by the *sighrakarna*, should be applied to the mean Sun. This gives the geocentric longitude of Venus.

In the following, we symbolically represent the above procedure for finding the geocentric longitude of an interior planet, say Mercury. Unlike the procedure used for the exterior planets, in the case of interior planets, four corrections are not prescribed. Only two corrections are given: (i) *mandasamskara* to obtain the *mandasphuta-graha* from the *madhyamagraha* and (ii) *sighrasamskara* to obtain the *sphutagraha* from the *mandasphuta-graha*. Let l_0 be the mean longitude of the planet at the given time. The *mandasphuta-graha* l_1, is given by

$$l_1 = l_0 + \sin^{-1}\left(\frac{r \sin M}{karna}\right),$$

where M is the *mandakendra* = mean longitude of the planet – longitude of its apogee, the ratio of the *manda* epicycle to the deferent circle is given by $\frac{r}{R} = \frac{1}{6}$, and

$$karna = \left[(R + r \cos M)^2 + (r \sin M)^2\right]^{1/2}.$$

The *sphutagraha* or geocentric longitude of the planet l_2 is obtained from l_1 by

$$l_2 = \textit{longitude of the mean Sun} + \sin^{-1}\left(\frac{r \sin S}{karna}\right),$$

where S is the *sighrakendra* = l_1 - longitude of its *sighroccha*, the ratio of the *sighra* epicycle to the deferent circle is given by

$$\frac{r}{R} = \left(\frac{31 - 2|\sin S|}{80}\right),$$

and

$$karna = \left[(R + r \cos S)^2 + (r \sin S)^2\right]^{1/2}.$$

The *mandasamskara* prescribed by Nilakantha for Mercury and Venus in the above verses is different from the ones given in the earlier astronomical texts. Here Nilakantha prescribes that the *mandaphala* or equation of centre has to be applied to the actual mean longitude of the interior planet, and not to the mean Sun as was the practise till then.

The ratios of the *manda* and *sighra* epicycles to the deferent circle, given in the above verses, for the interior planets Mercury and Venus, are listed in Table 3.

Thus we find Nilakantha to be the first savant in the history of astronomy to have arrived at the proper application of equation of centre in the case of interior planets. This is indeed a major landmark in the history of astronomy, since no astronomer prior to Nilakantha (c.1500), seems to have clearly visualized the motion of interior planets and arrived at the exact procedure for getting their geocentric longitudes and latitudes.

Ratio \ Planet	Mercury	Venus
$\frac{r}{R}$ (*Manda*)	$\frac{1}{6}$	$\frac{1}{14+\frac{R\vert\sin M\vert}{240}}$
$\frac{r}{R}$ (*Sighra*)	$\frac{31-2\vert\sin S}{80}$	$\frac{59-2\vert\sin S\vert}{80}$

Table 3: Ratio of *manda* and *sighra* epicycles to the deferent circle for the interior planets. R in the above table is referred as *trijya* whose value is taken to be 3438.

Unified procedure for the computation of planetary latitudes

By revising the traditional Indian planetary model, as regards the application of the equation of centre for the interior planets, Nilakantha arrived at the correct procedure for the computation of the longitudes of all the five planets. In this way he was also able to provide a unified procedure for the computation of the planetary latitudes. According to this procedure, the latitudes of all planets are computed from the *mandasphutagraha* or the true heliocentric longitude of the planet. The following verse in *Tantrasangraha* gives the procedure for the computation of planetary latitudes[16]:

> मन्दस्फुटात् स्वपातोनात् भौमादीनां भुजागुणात्।
> परमक्षेपनिघ्ना स्यात् क्षेपोऽन्त्यश्रवणोद्धृतः।।
>
> The sine of the difference in the longitudes of the nodes and the *mandasphutagraha* of Mars etc.(i.e., all the five planets) multiplied by the inclination of their own orbits and divided by the iteratively obtained hypotenuse, gives the latitude of the planet at the desired instant.

[16] *Tantrasangraha* VII.4-5.

Symbolically representing the procedure given in the above verse we have,

$$\beta = \frac{i\, R\sin(\lambda_m - \lambda_n)}{karna},$$

where β, i represent latitude (heliocentric) and the inclination of the orbit of the planet, λ_m refers to the *mandasphuta,* or the true heliocentric longitude of the planet ; λ_n refers to the longitude of the node, and

$$karna = \left[(R + r\cos M)^2 + (r\sin M)^2\right]^{1/2}.$$

What is noteworthy is that Nilakantha does not give two distinct formulae, one for the computation of the latitudes of the interior planets and one for the exterior ones, as found in the earlier texts. He provides a single formula using which the latitudes of all the five planets can be computed in a unified way.

Conversion of heliocentric latitude to geocentric latitude

Since Nilakantha has clearly recognized that the centre of the mean planet's orbit is the mean Sun and not the Earth, he has discussed the problem of conversion of heliocentric latitude to geocentric latitude. A very clear and detailed exposition of this is found in the *Aryabhatiyabhashya* of Nilakantha where he observes [17]:

ताराग्रहाणां पुनः शीघ्रोच्चनीचवृत्तं न विक्षिप्तम्। अपमण्डलानुसारिणि तत्परिधौ यत्र आदित्यसूत्रं स्पृशति तत्रैव पञ्चानां स्वस्वोच्चनीचवृत्तपरिधौ मन्दोच्चनीचवृत्तकर्णमण्डलयोर्मध्यम्। तद्द्वयं विक्षिप्तं चार्धशः अपमण्डलमभितः॥

> In the case of the planets (all the five), the *sighroccha-nicha-vritta* is not inclined. The centre of the *mandoccha-nicha-vritta* and the *mandakarnamandala* of all the five planets lie on the circumference of their own *sighroccha-nicha-vritta*s at the longitude of the Sun and which (*sighroccha-nicha-vrittas*) lie in the plane of the ecliptic. These two circles (*mandoccha-nicha-vritta* and the *mandakarnamandala*) are half-way deflected from the ecliptic.

Here, Nilakantha clearly mentions that the orbit in which planets move, *mandakarnamandala,* is centered around the mean Sun and that these orbits are inclined to the ecliptic. This is schematically depicted in Fig.3.

[17] *Aryabhatiyabhashya* Golapada. verse.3.

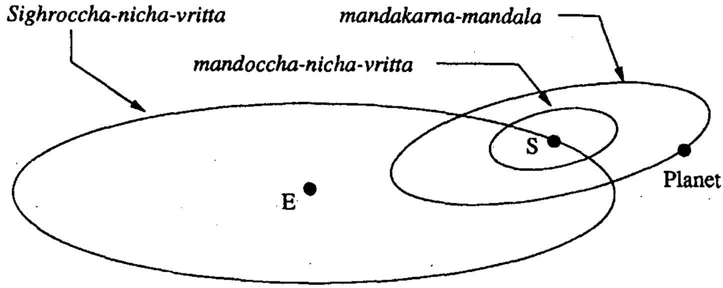

Fig. 3: *Sighroccha-nicha-vritta* (in which the Sun moves around the Earth), the *mandoccha-nicha-vritta* and *mandakarna-mandala* of an interior planet. Of these three circles, the latter two are inclined to the former, which lies in the plane of the ecliptic.

Nilakantha further states that the latitude obtained by using Eq.(3) is not the true geocentric latitude. Then he explains how to obtain the latter from the former as follows[18]:

कथं पुन: प्रकृतं भगोलकलाप्रमितविक्षेपानयनं; यदर्थं एतत् सर्वं प्रदर्शितम्; उच्यते। य इह पातोनमन्दस्फुटभुजां परमविक्षेपेण हत्वा त्रिज्याप्तं मन्दकर्णकलाप्रमितं विक्षेपमेव अविशिष्टमन्दकर्णेन हत्वा त्रिज्यया हृत्वा लब्धो मध्यकक्ष्याकलाप्रमितो विक्षेप:; तमेव व्यासार्धेन हत्वा विक्षेपकोट्यानीतशीघ्रकर्णवर्गे मध्यकलाप्रमितविक्षेपवर्गं संयोज्य पदीकृतेन भूताराग्रहविवरेण हरेत्। तत्र लब्ध: स्फुटविक्षेप:।

> How to obtain the measure of true (geocentric) latitude, for the sake of which all this discussion has been carried out? We will explain it now: The latitude which was obtained by multiplying the sine of the difference in longitude of the node and the *mandasphuta*, with the maximum inclination of the orbit and dividing by *trijya*, is indeed the latitude with respect to the measure of *mandakarna*. The same, multiplied by the iterated *mandakarna* and divided by the *trijya*, gives the latitude with respect to *madhyakakshya*. Let this latitude be multiplied by *trijya* and divided by the *bhutaragraha-vivira* (distance of seperation between the Earth and the planet) which is the square root of the sum of the squares of the *sighrakarna* and the latitude obtained with respect to the *madhyamakakshya*. The latitude thus obtained is the true (geocentric) latitude.

[18] *Aryabhatiyabhashya* Golapada. verse.3.

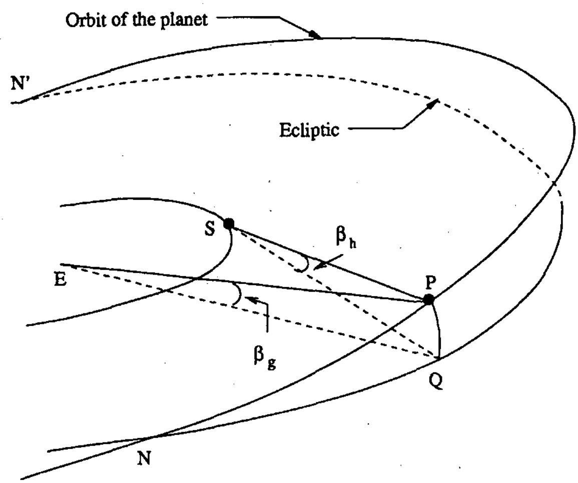

Fig. 4: Orbit of an outer planet P, and that of the Sun S, around the Earth E. This figure is used to find out the relation between the heliocentric latitude β_h and the geocentric latitude β_g of the planet.

We shall demonstrate the above procedure in the case of exterior planets. In Fig.4, let *N* and *N'* represent the nodes of the planetary orbit. *NPN'* and *NQN'* represent the planetary orbit and the ecliptic respectively. *E*, *S* and *P* represent the Earth, Sun and the planet respectively. The arc *PQ* which is the deflection of the planet from the plane of the ecliptic subtends two different angles β_h and β_g at the location of the Sun and Earth respectively. These angles β_h and β_g are the heliocentric and the geocentric latitudes of the planet respectively. The latitude one obtains from Eq.(3) is β_h, which is essentially equal to the angle subtended by the great circle *PQ* at the location of the Sun. In the following, we derive the expression for β_g in terms of β_h.

Though *PQ* in Fig.4. forms part of a great circle, since *PQ* is relatively small, it can be treated as a straight line to a good approximation. From the triangles *SPQ* and *EPQ*, it is easy to see that

$$SP \sin\beta_h = EP \sin\beta_g ,$$

or

$$\sin\beta_g = \frac{SP \sin\beta_h}{EP}. \qquad (5)$$

Since β_h and β_g are small, the above equation can be simply written as

$$\beta_g = \frac{SP\,\beta_h}{EP}. \tag{6}$$

This is exactly the relationship given by Nilakantha, where he refers to *SP* as *trijya* and *EP* as *bhutaragrahavivara*. This quantity *EP* which corresponds to the distance of separation between the Earth and the planet is stated to be equal to the square-root of the sum of the squares of the *sighrakarna* and the latitude *i.e.* $\sqrt{EQ^2 + PQ^2}$.

Adoption of Nilakantha's model by later astronomers

Nilakantha's revision of the traditional planetary theory was adopted by his pupils, Sankara Varier and Chitrabhanu and also by most of the later astronomers of the Kerala School, such as Jyesthadeva, Achyuta Pisharati and Putumana Somayaji. Chitrabhanu in his *Karanamrita*, while giving the procedure for finding the *madhyamagraha* or mean positions of the planets, states [19]:

चतुर्घ्नार्कोऽर्कषष्ठांशयुतस्सौम्यस्ततः पुनः।
शराभ्रेन्दुहतान्मासाद्वेदाप्ता लिप्तिकास्त्यजेत्।।
साध्यर्धार्केऽर्काष्टमांशे क्षिपेन्मासकलां भृगोः।।

Similarly, Achyuta Pisharati, in his *Karanottama*, states [20]:

चतुर्घ्नो भगणो दिघ्नद्युगणाष्टशतांशयुक्।
द्विनिघ्नद्युगणोऽग्न्यर्कभक्तलिप्तोनितो बुधः।।
दिनं षड्घ्नदिनांशांशयुतं शुक्रो दिनाद्गजैः।

As the names themselves suggest, *Karanottama* and *Karanamrta* are manuals used for computations (*karana* texts) which would simplify calculations. By using these *karana* texts one avoids dealing with large numbers that would be involved in the computations otherwise. As we are not discussing here the details of the procedures given by Chitrabhanu or Achyuta, we do not provide a translation or explanation of the above passages. However we would like to emphasise that the mean planet in

[19] *Karanamrita* I. 6-8.
[20] *Karanottama* I. 6-8.

the case of Mercury and Venus is not taken to be the mean Sun, by both these authors. This is clear from the fact that they have given two different formulae for computing the mean positions of Mercury and Venus.

While most of the later Kerala astronomers have adopted the theory proposed by Nilakantha, some of them have reformulated the *sighra samskara* given in *Tantrasangraha* for interior planets in a form analogous to the one employed for exterior planets. There are perhaps many other improvements carried out by them which are yet to be brought to light.

Geometrical Picture of Planetary Motion According to Nilakantha

M.D. Srinivas

Modern scholars of Indian astronomical tradition have noted that the Indian astronomers were mainly interested in the successful computations of the longitudes and latitudes of the Sun, Moon and the planets, and were not much concerned about proposing models of the universe. The Indian astronomical texts, as a rule, present detailed computational schemes for calculating the geocentric positions of the Sun, Moon and the planets. Their exposition of planetary models is by and large analytical and the geometrical picture of planetary motion does not play any crucial role in their basic formulations.

Sometimes, the Indian texts of astronomy also include a discussion of the geometrical picture of planetary motion as implied by their computational schemes. This tradition is especially noticeable amongst the Kerala School of Astronomers. Paramesvara of Vatasseri (1380-1460 AD), the *Paramaguru* of Nilakantha, presented a detailed exposition of the geometrical picture of planetary motion as implied by the traditional planetary model employed by the Indian astronomers, at least since the time of Aryabhata (499 AD). Following this, Nilakantha (1444-1545 AD) discussed the geometrical picture of planetary motion that is implied by his own revised planetary model. According to Nilakantha, the five Planets – Mercury, Venus, Mars, Jupiter and Saturn – move in eccentric orbits around the mean Sun, which in turn goes around the Earth.[1]

The geometrical picture of planetary motion as outlined by Nilakantha does seem similar to the model of planetary motion which was proposed nearly a century later by the European astronomer Tycho Brahe (c.1583). However, Nilakantha's fairly accurate understanding of the geometrical orbits of the planets does not arise in the course of any debate concerning heliocentric and geocentric cosmologies. Indeed, the outstanding achievements of Nilakantha and Tycho Brahe belong to different traditions

[1] This fact was first noticed in K.Ramasubramanian, M.D.Srinivas and M.S.Sriram, Curr. Sc. **66**, 784, 1994.

of astronomy. The motivation and the spirit behind their seemingly identical geometrical models of planetary motion, and the way they arrive at them, all seem to be profoundly different. To understand the work of Nilakantha in the proper perspective it is essential to have some idea of the crucial differences in approach between the Greeko-European tradition in Astronomy and the Indian tradition in Astronomy, especial ly as regards planetary theory.

The Greek approach to planetary theory as expounded in Ptolemy's Almagest

One of the best sources to study the Greek approach to planetary theory is the great work of Claudius Ptolemy (c.150 AD), *The Mathematical Syntaxis*, more popularly known by its Arabic name, *The Almagest*, which contains the most systematic exposition of Greek mathematical astronomy. In the first section of *The Almagest*, Ptolemy summarises the Aristotelian classification of natural philosophy into physics, mathematics and theology. Of these, physics, which dealt with the "corruptible bodies...below the lunar sphere", could never be an exact discipline worthy of philosophers' attention; and theology, which dealt with "the first cause of the first motion of the universe ...is completely separated from perceptible reality." Only mathematics, which concerned itself with "eternal things with an ethereal nature", or "divine and heavenly things", namely the celestial objects above the lunar sphere, could provide "sure and unshakeable knowledge to its devotees". In Ptolemy's own words:[2]

> For Aristotle divides theoretical philosophy too, very fittingly, into three primary categories, physics, mathematics and theology. For everything that exists is composed of matter, form and motion; none of these [three] can be observed in its substratum by itself, without the others: they can only be imagined. Now the first cause of the first motion of the universe, if one considers it simply, can be thought of as an invisible and motionless deity; the division [of theoretical philosophy] concerned with investigating this [can be called] 'theology', since this kind of activity, somewhere up in the highest reaches of the universe, can only be imagined, and is completely separated from perceptible reality. The division [of theoretical philosophy] which investigates material and ever-moving nature, and which concerns itself with 'white', 'hot', 'sweet', 'soft' and suchlike qualities one may call

[2] *The Almagest* of Claudius Ptolemy, Tr. G. J. Toomer, Duckworth, London, 1984. p. 36-7.

> 'physics'; such an order of being is situated (for the most part) amongst corruptible bodies and below the lunar sphere. That division [of theoretical philosophy] which determines the nature involved in forms and motion from place to place, and which serves to investigate shape, number, size, and place, time and suchlike, one may define as 'mathematics'. Its subject-matter falls as it were in the middle between the other two, since, firstly, it can be conceived of both with and without the aid of the senses, and, secondly, it is an attribute of all existing things without exception, both mortal and immortal: for those things which are perpetually changing in their inseparable form, it changes with them, while for eternal things which have an ethereal nature, it keeps their unchanging form unchanged.
>
> From all this we concluded: that the first two divisions of theoretical philosophy should rather be called guesswork than knowledge, theology because of its completely invisible and ungraspable nature, physics because of the unstable and unclear nature of matter; hence there is no hope that philosophers will ever be agreed about them; and that only mathematics can provide sure and unshakeable knowledge to its devotees, provided one approaches it rigorously. For its kind of proof proceeds by indisputable methods, namely arithmetic and geometry. Hence we are drawn to the investigation of that part of theoretical philosophy, as far as we were able to the whole of it, but especially to the theory concerning the divine and heavenly things. For that alone is devoted to the investigation of the eternally unchanging. For that reason it too can be eternal and unchanging (which is a proper attribute of knowledge) in its own domain, which is neither unclear nor disorderly.

In the third section of Book I of *The Almagest*, Ptolemy goes on to explain that the celestial bodies, being constituted of the ideal substance ether, are endowed with the ideal shape, namely that of a sphere; they undergo only ideal motion, namely uniform circular motion: [3]

> The ether is, of all bodies, the one with constituent parts which are finest and most like each other; now bodies with parts like each other have surfaces with parts like each other; but the only surfaces with parts like each other are the circular, among the planes, and the spherical among the three-dimensional surfaces. And since the ether is not plane, but three-dimensional, it follows that it is spherical in shape. Similarly, nature formed all Earthly and corruptible bodies out of shapes which are round but of unlike parts, but all ethereal and divine bodies out of shapes which are of like parts and spherical. For if they were flat or shaped like a discus they would not always display a circular shape to all those observing them from simultaneously from different places on Earth. For this reason it is plausible

[3] *The Almagest,* cited above, p. 40.

> that the ether surrounding them, too, being of the same nature, is spherical, and because of the likeness of its parts moves in a circular and uniform motion.

Ptolemy takes up the subject of planetary motion in Book IX of *The Almagest*. In the second section he enunciates the basic hypothesis that their motion, like that of the Sun and the Moon, ought to be "represented by uniform circular motions", as that is what is proper for these "divine beings". In Ptolemy's words:[4]

> Now it is our purpose to demonstrate for the five planets, just as we did for the sun and moon, that all their apparent anomalies can be represented by uniform circular motions, since these are proper to the nature of divine beings, while disorder and non-uniformity are alien [to such beings]. Then it is right that we should think success in such a purpose a great thing, and *truly the proper end of mathematical part of theoretical philosophy. But, on* many grounds, we must think that it is difficult, and there is good reason why no one before us has yet succeeded in it...
>
> Hence it was, I think, that Hipparchus, being a great lover of truth, for all the above reasons, and especially because he did not yet have in his possession such a ground-work of resources in the form of accurate observations from earlier times as he himself has provided to us, although he investigated the theories of the sun and moon, and, to the best of his ability, demonstrated with every means at his command that they are represented by uniform circular motions, did not even make a beginning in establishing theories for the five planets, not at least in his writings which have come down to us. All that he did was to make a compilation of the planetary observations arranged in a most useful way, and to show by means of these that the phenomena were not in agreement with the hypotheses of the astronomers of that time.

To some extent the above extracts from *Almagest* summarise the basic approach to astronomy that prevailed in the Greeko-European tradition till nearly the end of sixteenth century.

The Indian approach to planetary theory

The Indian texts of Astronomy, or *Jyotihsastra*, present as the main *prayojana* or the purpose of the *sastra* to be the determination of *kala*

[4] *The Almagest*, cited above, p. 420-1. Incidentally, at this point, Ptolemy also declares that his was the first serious attempt to develop a planetary theory (in the Greek tradition), as even the celebrated Hipparchus (130 BC) made no headway in this subject.

(time), *dik* (direction) and *desa* (place). The ancient *Vedangajyotisha* texts declare *Jyotihsastra* to be *kalavidhana sastra*, the science of determining time. One of the standard texts of *Jyotihsastra*, *Siddhantasiromani* of Bhaskaracharya (c. 1150 AD) states that "from this [*Jyotih*]*sastra* there arises *kalabodha*, the knowledge of time". And, his commentator Nrisimha Daivajna (c.16th Century) explains "that the term *kala* also encompasses *dik*".[5] Now, the determination of *kala*, *dik* and *desa* is to be achieved through *grahagati-pariksha*, a study of the motion of the celestial objects.

Thus, the object of *Jyotihsastra* was not to discover the true cosmological model of the universe, or even the true laws of planetary motion; it was the more practical one of determining time, direction and space accurately by a careful study of the motion of the celestial bodies. For this purpose, the Indian astronomers put all efforts in making accurate observations, developing suitable theories and efficient methods of calculation, and evolving critical tests to help them correct their theories whenever their calculations failed to correspond with the observation.

Further, the Indian Astronomical texts repeatedly emphasise that *sastras* become *slatha*, inadequate and weak, over time. This is taken to be inherent in the very nature of things, although, sometimes, detailed reasons are given as to why many great *sastras* of ancient times have become inadequate. The indication that the *sastra* has become *slatha* is almost always found in the failure to achieve *drigganitaikya*, concordance between calculation and observation. And whenever a *sastra* becomes *slatha*, the Astronomers are expected to undertake *sastra-samsthapana*, reestablishment of the *sastra* by careful re-examination of their theories leading to revision of the various procedures and parameters used in them. Many a times this would have proved to be too daunting a task. Commenting on the faint-heartedness of some of his predecessors, Nilakantha declares in his seminal work *Jyotirmimamsa*:[6]

मानसव्याख्यातापि कश्चिदाह - 'ननु पैतामहादिभेदेन परस्परविरुद्धाश्च सिद्धान्ता भवन्ति। सिद्धान्तभेदे सति कालभेदः। कालभेदे सति कालांगानि श्रौतस्मार्तलौकिकानि कर्माणि विकलानि स्युः। कर्मवैकल्ये सति लोकयात्रोच्छेदः। हा धिक्! संकटे महति पतिताः स्मः'।

अत्रोच्यते - ऋजुमते! स न शोचितव्यः। गुरुचरणपरिचरणपरैः किमिव न ज्ञायते।

[5] *Siddhantasiromani* of Bhaskaracharya, with *Vasanabhashya* and *Vasanavarttika* of Nrisimha Daivajna, Ed. Muralidhara Chaturvedi, Varanasi, 1981, p.10-11.

[6] *Jyotirmimamsa* of Nilakantha, Ed. K.V.Sarma, Hoshiarpur, 1977, p.6

पञ्चसिद्धान्तास्तावत् क्वचित्काले प्रमाणमेव इत्यवगन्तव्यम्। अपि च यः सिद्धान्तो दर्शनाविसंवादी भवति सोऽन्वेषणीयः। दर्शनसंवादश्च तदानीन्तनैः परीक्षकैर्ग्रहणादौ विज्ञातव्यः। ये पुनरन्यथा, प्राक्तनसिद्धान्तस्य भेदे सति, यन्त्रैः परीक्ष्य ग्रहाणां भगणादिसंख्यां ज्ञात्वा अभिनवसिद्धान्तः प्रणेय इत्यर्थात्।

> A commentator on the *Manasa* [*Laghumanasa* of *Munjala*] has lamented: "Indeed, the *siddhantas*, like *Paitamaha*, differ from one another [in giving the astronomical constants]. Timings are different as the *siddhantas* differ [i.e. the measures of time at a particular moment differ as computed by the different *siddhantas*]. When the computed timings differ, Vedic and domestic rituals, which have [correct] timings as a component [of their performance] go astray. When rituals go astray, worldly life gets disrupted. Alas, we have precipitated into a calamity.
>
> Here, it needs to be stated: "O faint-hearted, there is nothing to be despaired of. Wherefore does anything remain beyond the ken of those intent on serving at the feet of the teachers [and thus gain knowledge]? One has to realise that the five *siddhantas* had been correct at a particular time. Therefore, one should search for a *siddhanta* that does not show discord with actual observations [at the present time]. Such accordance with observation has to be ascertained by [astronomical] observers during times of eclipses etc. When *siddhantas* show discord, i.e., when an early *siddhanta* is in discord, observations should be made of revolutions etc. [which would give results, which accord with actual observation] and a new *siddhanta* enunciated.

In the same work, Nilakantha then goes on to an interesting discussion about reconciling the common understanding that all *sastras* are in some sense 'divine revelations' and the inexorable fact that a *sastra* needs to be revised over time. He says that it is true that all *sastras* are divinely inspired and *sastra-samsthapana* cannot be done without divine grace; but *sastras* remain essentially human creations, though of gifted individuals, and cannot be expected to be the ultimate or absolute truth. Therefore, the *sastras,* even though they are creation of great and inspired individuals, need to be constantly put to test and revised if necessary.

It is this understanding of the *sastra* as an essentially human construct (*purusha-buddhi-prabhava*) that enables Indian scientists to reconcile and live with several schools of thought (*siddhantas* or *pakshas*) in any *sastra* as long as the *prayojana* of the *sastra* is achieved. If the purpose of *Jyotihsastra* were to arrive at the true picture of the heavens, then when Aryabhata proposed the model of diurnal rotation of Earth as opposed to the (then) traditional model of the rotation of the celestial sphere, all work in *Jyotihsastra* would have focused only on resolving

which of the two models was the 'true' one. Instead, Indian astronomers of both schools continued to concentrate on refining basic astronomical parameters and computational schemes in order to arrive at better accord with observations. Settling what constitutes a true picture of the world was surely not the *raison-de- etre* of their science.

As regards the epistemological status of the planetary models, the Indian astronomical texts present a very clear position that they are conceptual tools which serve the purpose of calculating observationally verified planetary positions. Notions such as the apsides (*uccha, nicha*), mean (*madhyama*), eccentrics or epicycles used in *manda* and *sighra* corrections (*manda-paridhi* etc.) and so on – notions which are employed in various planetary models – are all conceptual constructs and there are no constraints on our choice of them except that the model should lead to results in concordance with observations. This position is clearly set out for instance in the famous *Aryabhatiyabhashya* of Bhaskaracharya I (c. 629 AD), when he starts his exposition of the *manda* and *sighra* corrections:[7]

उच्च-नीच-मध्यम-परिधिरित्येवमादि-स्फुटगतिसाधनोपाय (भूतानाञ्च) उपायानां नैव नियमोक्तिर्वा विद्यते। केवल तु उपेयसाधका उपाया:। तस्मादिय सर्वा प्रक्रिया असत्या, यया ग्रहाणा स्फुटगति: साद्ध्यते। (एव च परमार्थजिज्ञासुभि: असत्योपायेन) सत्यं प्रतिपद्यते। तथाहि - भिषजो ह्युत्पलनालादिषु वेधादीन्यभ्यस्यन्ते, नापिता: पिठरादिषु मुण्डनादीनि, यज्ञशास्त्रविद: शुष्केष्ट्या यज्ञादीनि, शाब्दिका: प्रकृतिप्रत्ययविकारागम-वर्णलोपव्यत्ययादिभि: शब्दान् प्रतिजानते। एवमत्रापि मध्यम-मन्दोच्च-शीघ्रोच्च-तत्परिधि-ज्या-काष्ठ-भुजा-कोटि-कर्णादिव्यवहारेण सावत्सरा ग्रहाणा स्फुटगतिं प्रतिजानते। तस्मादुपायेष्वसत्येषु सत्यप्रतिपादनपरेषु न चोद्यमस्ति।

> There are no constraints or limitations imposed on the notions such as the *uccha*, *nicha*, *madhyama*, *paridhi* and so on which are indeed aids to the calculation of the observed motion of the planets. These are only the means for arriving at the desired results. Hence this entire procedure is fictitious, by means of which the observed planetary motion is arrived at. Just as the seekers of ultimate knowledge expound the ultimate truth via untrue means; just as the surgeons practice their surgery etc on stems and other objects; just as the hair-stylists practice shaving on pots; just as the experts in performance of *yajna* practice using dry wood; just as the linguists utilise notions such as *prakriti*, *pratyaya*, *vikara*, *agama*, *varna*, *lopa*, *vyatya*, etc., to comprehend (well formed) words; in the same way in our science also the astronomers employ notions such as *madhyama*, *mandoccha*,

[7]*Aryabhatiyabhashya* of Bhaskaracharya I, Ed. K.S. Shukla, New Delhi, 1976, p.217.

> *sighroccha, sighra-paridhi, jya, kashtha, bhuja, koti, karna*, etc., in order to comprehend the observed motion of planets. Hence, there is indeed nothing unusual that fictitious means are employed to arrive at the true state of affairs [in all these sciences].

There is a very similar statement made by the renowned Astronomer Chaturveda Prithudakasvamin (c. 864 AD) in his celebrated commentary on *Brahmasphutasiddhanta* of Brahmagupta:[8]

तथा चाह ब्रह्मगुप्तभाष्यकारश्चतुर्वेदाचार्यः-
यथा वैयाकरणाः प्रकृतिप्रत्ययागमलोपविकारैः असत्यरूपैः सत्यं शब्दसाधुत्वं प्रतिपद्यन्ते, यथा च भिषग्वरा उत्पलनालादिभिः शिरावेधादीन् प्रतिपद्यन्ते, तथैव सांवत्सराः फलावलम्बमन्दशीघ्रप्रतिमण्डलादिभिः ग्रहगतितत्त्वं भूमानादितत्त्वञ्च प्रतिपद्यन्त इति मत्वा सन्तोष्टव्यमिति।

> Just as the grammarians employ fictitious entities such as *prakriti, pratyaya, agama, lopa, vikara*, etc. to decide on the established real word forms, and just as the *vaidyas* employ tubers etc. to demonstrate surgery, one has to understand and feel contented that it is in the same way that the astronomers postulate measures of the Earth etc. and models of motion of the planets in *manda* and *sighra-pratimandalas* for the sake of accurate predictions.

In his *Aryabhatiyabhashya*, Nilakantha also repeats the same epistemological principle that there are indeed no constraints or requirements that need to be imposed on theoretical models or procedures except that they have to lead to valid results. He goes on to quote the famous verse of Bhartrihari's *Vakyapadiya* which propounds this view:[9]

उपेयस्यैव नियमः नोपायानाम्।
उपादेया न ये हेयास्तानुपायान् प्रचक्षते। उपायानां च नियमो नावश्यमवतिष्ठते।।
इत्युपायानामनियमः प्रकीर्णकेऽप्युक्तः।

The above discussion should make it amply clear that the Indian astronomers adopted an extraordinarily flexible and pragmatic view on the nature and purpose of planetary models. Further they were not constrained by any metaphysical presuppositions regarding the celestial bodies or the ideal motions that they ought to follow. The Indian astronomical tradition was informed with the understanding that the

[8] *Vasanabhashya* of Prithudakasvamin on *Brahmasphutasiddhanta* of Brahmagupta, cited by Nrisimha Daivajna in his *Vasanavarttika* on Bhaskaracharya's *Siddhantasiromani*, cited above, p.48.

[9] *Aryabhatiyabhashya* of Nilakantha, *Kalakriyapada*, Ed. K.Sambasiva sastri, Thiruvanantapuram, 1931, p.41.

motions of the heavenly bodies are fairly complex, and allowed for a fairly high degree of flexibility and sophistication in the computational schemes that were to be employed for describing the planetary motions. These computational schemes were presented in an analytical manner, but many steps involved had fairly simple geometrical interpretation. Such geometrical interpretations were frequently presented in the Indian astronomical texts, but there was often the cautionary note that reality is far more complex than implied by such simple geometrical pictures. This is in marked contrast with the kind of approach that is characteristic of most of the developments in the Greeko-European tradition of astronomy till the modern times.

Geometrical picture of planetary motion according to Paramesvara

The renowned Kerala astronomer Paramesvara of Vatasseri (1380-1460) has discussed in detail the geometrical model implied in the conventional planetary model of Indian Astronomy. Damodara the son and disciple of Paramesvara was the teacher of Nilakantha. Nilakantha often refers to Paramesvara as *Paramaguru*.

In his super-commentary *Siddhantadipika* (on Govindasvamin's commentary on) *Mahabhaskariya* of Bhaskaracharya-I, Paramesvara gives a detailed exposition of the geometrical picture of planetary motion.[10] A shorter version is available in his commentary on *Aryabhatiya*, which is given below:[11]

स्फुटविधियुक्तिस्सिध्येन्नैव विना छेद्यकेन विहगानाम्।
तस्मादिह संक्षेपाच्छेद्यककर्म प्रदर्श्यते तेषाम्।।

त्रिज्याकृतं कुमध्यं कक्ष्यावृत्तं भवेत्तु तच्छैघ्रम्।
शीघ्रदिशि तस्य केन्द्रं शीघ्रान्त्यफलान्तरे पुनः केन्द्रम्।।

कृत्वा विलिखेद्वृत्तं शीघ्रप्रतिमण्डलाख्यमुदितमिदम्।
इदमेव भवेन्मान्दे कक्ष्यावृत्तं पुनस्तु तत्केन्द्रात्।।

[10] *Siddhantadipika* of Paramesvara on *Mahabhaskariyabhashya* of Govindasvamin, Ed. T.S. Kuppanna Sastri, Madras, 1957, p.233-238.

[11] *Bhatadipika* of Paramesvara on *Aryabhatiya*, Ed. H. Kern, Leiden, 1874, p.60-1. It is surprising that this work, published over 125 years ago, has not received any serious scholarly attention.

केन्द्रं कृत्वा मन्दान्त्यफलान्तरे वृत्तमपि च मन्ददिशि।
कुर्यात्प्रतिमण्डलमिदमुदितं मान्दं शनीड्यभूपुत्राः।।

मान्दप्रतिमण्डलगास्तत्कक्ष्यायां तु यत्र लक्ष्यन्ते।
तत्र हि तेषां मन्दस्फुटाः प्रदिष्टास्तथैव शैघ्रे ते।।

प्रतिमण्डले स्थितास्स्युस्ते लक्ष्यन्ते पुनस्तु शैघ्राख्ये।
कक्ष्यावृत्ते यस्मिन् भागे तत्र स्फुटग्रहास्ते स्युः।।

एवं सिध्यति तत्र स्फुटयुग्मं तत्र भवति दृग्भेदः।
यत्र खगा लक्ष्यन्ते तत्रस्था लक्षिता यतोऽन्यस्मिन्।।

क्रियतेऽत्र तन्निमित्तं मध्ये मान्दार्धमपि च शैघ्रार्धम्।
शैघ्रं मान्दं मान्दं शैघ्रञ्चेति क्रमस्मृतोऽन्यत्र।।

मान्दं कक्ष्यावृत्तं प्रथमं बुधशुक्रयोः कुमध्यं स्यात्।
तत्केन्द्रान्मन्ददिशि मन्दान्त्यफलान्तरे तु मध्यं स्यात्।।

मान्दप्रतिमण्डलस्य तस्मिन् यत्र स्थितो रविस्तत्र।
प्रतिमण्डलस्य मध्यं शैघ्रस्य तस्य मानमपि च गदितम्।।

शीघ्रस्ववृत्ततुल्यं तस्मिंश्चरतस्सदा ज्ञशुक्रौ च।
स्फुटयुक्तिः प्राग्वत्स्याद्दृग्भेदः पूर्ववद्भवेदिह च।।

क्रियतेऽत्र तन्निमित्तं शैघ्रार्धं व्यत्ययेन मन्दोच्चे।
तत्सिद्धं मान्दं प्राक् पश्चाच्छैघ्रञ्च सूरिभिः पूर्वैः।।

Since the rationale for the *sphutavidhi* (the scheme of computing the true planet) for the celestial bodies is not clear without the aid of *chedyaka* (diagrams), we present briefly the way of obtaining the diagrams.

For Mars, Jupiter and Saturn, with the centre of the Earth as the centre, the *sighra kakshya-vritta* (concentric circle) is drawn with *trijya* (Rsin90) as the radius. Then draw the *sighra pratimandala* (eccentric circle) with centre located at a distance equal to the *sighra-antya-phala* (maximum *sighra* correction) in the direction of *sighroccha*. The same will be the *manda* concentric. From its centre go along the direction of *mandoccha* a distance equal to the maximum *manda* correction and with this as centre draw a circle. This is referred as the *manda* eccentric circle. The planets Mars, Jupiter and Saturn move in this eccentric and when reduced to the *manda* concentric they are referred to as *mandasphuta,* and when reduced to the *sighra* concentric they are *sphuta* (true planets). But the true planets

obtained following this procedure, do not tally with the observed planets. Hence it is that, in the *manda* process, half the *manda* correction and half the *sighra* correction are applied. The sequence *sighra, manda, manda* and *sighra* is enunciated elsewhere.

For Mercury and Venus, the *manda* concentric circle is first drawn with the centre of the Earth as the centre. From that go along the direction of *mandoccha* a distance equal to the maximum *manda* correction and with that as the centre draw the *manda* eccentric circle. The point where the Sun is located on that eccentric is the centre of the *sighra* epicycle and the radius of that circle is [not *trijya* but] as enunciated. In that *sighra* epicycle, the Mercury and Venus always move. As in the case of the exterior planets, the true planets obtained thus are not in accord with observations and hence the ancient savants state that the *mandoccha* should be reduced by half of the *sighra* correction and then the *manda* correction should be done followed by the *sighra* correction.

The *chedyaka* procedure enunciated by Paramesvara is illustrated in Figures 1, 2. It is important to note that through his diagrammatic procedure, Paramesvara clearly illustrates the fact that in the traditional planetary model, the final longitude that is calculated for an interior planet is actually the geocentric longitude of what is called the *sighroccha* of the planet.[12] From Figs. 1, 2, we can see that Paramesvara's geometrical picture of planetary motion is fairly accurate, except for the fact that the equation of centre for the interior planets is wrongly applied to the mean Sun .

The above extract from *Bhatadipika* of Paramesvara brings out several important aspects that are characteristic of the Indian astronomical tradition. For instance, right at the outset Paramesvara declares that the *chedyakavidhi*, diagrammatic procedure is needed to elucidate the rationale of the process of calculating the *sphutagraha*, or the so-called "true planet". It is not to be confused with any true representation of the planetary motion. Indeed, later in the discussion, Paramesvara also makes clear that this geometric picture is only approximate. Since the

[12] Towards the end of his exposition of the *chedyakavidhi* in his super commentary *Siddhantadipika* (cited above), Paramesvara suggests that what has been called the *sighroccha* of an interior planet in conventional planetary model, should be identified as the planet itself and the mean Sun should be taken as the *sighroccha* for all the planets, while computing the *sighra* correction. He also seems to imply that the Earth is not circumscribed by the orbit of the interior planets. Thus, many of the basic ideas which were used by Nilakantha in formulating his new model seem to be present already in the work of Paramesvara.

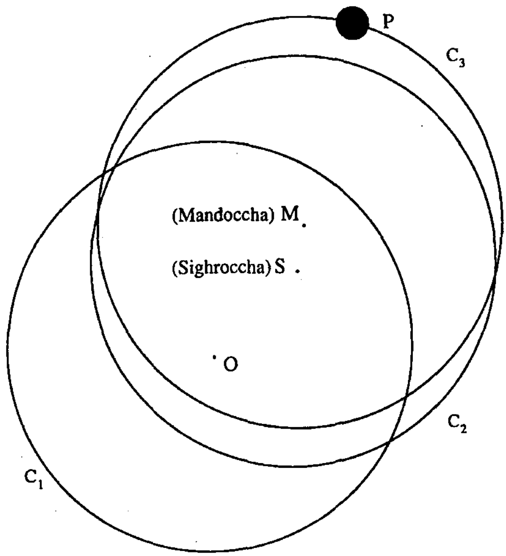

Fig. 1: Geometrical picture of the motion of an exterior planet as given by Paramesvara in his *Bhatadipika*. The circles C_1, C_2 and C_3 represent *Sighra-kakshyamandala*, *Sighra-pratimandala* (same as *Manda-kakshyamandala*) and *Manda-pratimandala* respectively. P is the planet and O is the centre of the Earth. OS and SM represent *Sighrantya-phala* and *Mandantya-phala* respectively.

sphutagrahas or the "true planets" calculated this way do not tally with the observed planets, the computational schemes suggested in Indian astronomical tradition involve correcting the *mandoccha* of the planets with half-*manda* and half-*sighra* corrections in the case of exterior planets and so on.

Paramesvara is clear that the actual computational scheme employed is too complex to be illustrated by simple geometrical pictures, involving sequence of concentric and eccentric circles. Paramesvara has also given a succinct description of the same *chedyakavidhi* in his *Goladipika*, which is given below:[13]

[13] *Goladipika* of Paramesvara, Ed. T. Ganapati Sastri, Thiruvanantapuram, 1916, p.14-15.

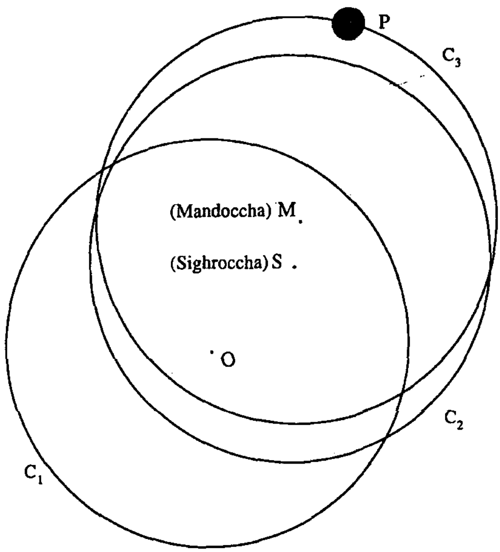

Fig. 2: Geometrical picture of the motion of an interior planet as given by Paramesvara in his *Bhatadipika*. The circles C_1, C_2 and C_3 represent *Manda-kakshyamandala*, *Manda-pratimandala* and *Sighraprati-mandala* respectively. P is the planet and O is the centre of the Earth. OM represents the *Mandantya-phala* and S is the Sun on *Manda-pratimandala*.

कर्णस्थितिसिद्ध्यर्थं स्फुटसिद्ध्यर्थं च लिख्यतेऽत्रापि।
कक्ष्यात्रयं झषान्ते प्राचीदिक् भवति सर्ववृत्तेषु।।
भूमध्यकेन्द्रमाद्यं भाख्यं वृत्तं तु भवति सर्वेषाम्।
तन्मध्याच्छीघ्रदिशि स्वान्त्यफलान्ते कुजार्यमन्दानाम्।।
शैघ्रस्य केन्द्रमुदितं बुधभृग्वोर्मन्ददिशि तु मान्दस्य।
स्वान्त्यफलान्ते केन्द्रद्वितीयमध्यात् कुजादीनाम्।।
मन्ददिशि मान्दकेन्द्रं द्वितीयपरिधिस्थभानुकेन्द्रमथ।
शैघ्रं ज्ञशुक्रयोः स्यादन्त्ये वृत्ते चरन्ति सर्वेऽपि।।

It is this succinct statement which has been taken as the model by Nilakantha, while enunciating his version of the geometrical picture of planetary motion as per his revised planetary model in *Siddhanta-darpana* and *Golasara*.

Qualitative picture of planetary motion according to Nilakantha

Like Paramesvara, Nilakantha is very much aware that the geometrical picture of planetary motion crucially depends on the computational scheme employed for calculating the planetary positions. In his *Aryabhatiyabhashya*, Nilakantha clearly explains that the orbits of the planets, and the various auxiliary figures such as the concentric and eccentric circles associated with the *manda* and *sighra* processes, are to be inferred from the computational scheme for calculating the *sphutagraha* (geocentric longitude) and *vikshepa* (latitude of the planets).[14]

Nilakantha's revision of the traditional computational scheme for the longitudes and latitudes of the interior planets, Mercury and Venus, was based on his clear understanding of the latitudinal motion of these planets. It is this understanding which also leads him to a correct qualitative picture of the motion of the inner planets. The best exposition of this by Nilakantha is to be found in his *Aryabhatiyabhashya*, which is reproduced below:[15]

अथ बुधशुक्रयोः भ्रमणप्रकारं तद्देशविशेषं च दर्शयति ... नन्वेवं बुधस्य द्वाविंशत्यैवाहोरात्रैर्वर्धमानस्य विक्षेपस्य महत्त्वनिवृत्तिः स्यात्, पुनर्ह्रासवशात् तावद्भिरेव दिनैः शून्यतापि स्यात्। एवमपक्रममण्डलादेकपार्श्वे एव गमनं चतुश्चत्वारिंशद्दिनान्येव, पुनरितरपार्श्वे तावन्त्येव दिनानि गच्छति। एवमष्टाशीत्यैव दिनैः विक्षेपस्य एकः पर्यायः परिसमाप्तः स्यात्, यतः अष्टाशीत्यैव दिनैः शीघ्रभगणपरिपूर्तिः।

शीघ्रवशाच्च विक्षेप उक्तः। कथमेतद्युज्यते। ननु स्वबिम्बस्य विक्षेपः स्वभ्रमणवशादेव भवितुमर्हति। न पुनरन्यभ्रमणवशादिति। सत्यम्। न पुनरन्यस्य भ्रमणवशादन्यस्य विक्षेपः उपपद्यते। तस्माद्बुधः अष्टाविंशत्यैव दिनैः स्वभ्रमणवृत्तं पूरयति ...एतच्च नोपपद्यते, यत एकेनैव संवत्सरेण तत्परिभ्रमणमुपलभ्यते, नैवाष्टाशीत्या दिनैः। सत्यम्। भगोळपरिभ्रमणं तस्याप्येकेनैवाब्देन... शुक्रोऽपि दिनानां पञ्चविंशत्यधिकशतद्वयेन स्वभ्रमणवृत्तं पूरयति। तयोः पातश्च तत्कर्णमण्डले कक्ष्यामण्डले वा वर्तते।

एतदुक्तं भवति–तयोर्भ्रमणवृत्तेन न भूः कबलीक्रियते। ततो बहिरेव सदा भूः। भगोळैकपार्श्वे एव तद्वृत्तस्य परिसमाप्तत्वात् तद्भगणेन न द्वादशराशिषु चारः स्यात्। तयोरपि वस्तुतः आदित्यमध्यम एव शीघ्रोच्चम्। शीघ्रोच्चभगणत्वेन पठिता एव स्वभगणाः। तथाप्यादित्यभ्रमणवशादेव द्वादशराशिषु चारः स्यात्... यथा कुजादीनामपि शीघ्रोच्चं स्वमन्दकक्ष्यामण्डलादिकमाकर्षति, एवमेवैतयोरपि।

[14] *Aryabhatiyabhashya* of Nilakantha, *Kalakriyapada*, cited above, p.70.

[15] *Aryabhatiyabhashya* of Nilakantha, *Golapada*, Ed. S. K. Pillai, Thiruvanantapuram, 1957, p. 8-9.

> Now he [Aryabhata] explains the nature of the orbits and their locations for Mercury and Venus... In this way, for Mercury the increase of the latitude occurs only for 22 days and then in the same number of days the latitude comes down to zero. Thus Mercury moves on one side of the *apamandala* (the plane of the ecliptic) for 44 days and it moves on the other side during the next 44 days. Thus one complete period of the latitudinal motion is completed in 88 days only, as that is the period of revolution of the *sighroccha* [of Mercury].
>
> The latitudinal motion is said to be due to that of the *sighroccha*. How is this appropriate? Isn't the latitudinal motion of a body dependent on the motion of that body only, and not because of the motion of something else? The latitudinal motion of one body cannot be obtained as being due to the motion of another body. Hence [we should conclude that] Mercury goes around its own orbit in 88 days... However this also is not appropriate because we see it going around in one year and not in 88 days. True; the period in which Mercury completes one full revolution around the *bhagola* (the celestial sphere) is one year only [like the Sun]... In the same way Venus also goes around its orbit in 225 days only...
>
> All this can be explained thus: The orbits of Mercury and Venus do not circumscribe the Earth. The Earth is always outside their orbit. Since their orbit is always confined to one side of the [geocentric] celestial sphere, in completing one revolution they do not go around the twelve *rasis* (the twelve signs). For them also really the mean Sun is the *sighroccha*. It is their own revolutions, which are stated to be the revolutions of the *sighroccha* [in ancient texts such as the *Aryabhatiya*]. It is only due to the revolution of the Sun [around the Earth] that they [i.e. the interior planets, Mercury and Venus] complete their movement around the twelve *rasis* [and complete their revolution of the Earth]... Just as in the case of the exterior planets (Jupiter etc.), the *sighroccha* (i.e., the mean Sun) attracts [and drags around] the *manda-kakshya-mandala* (the *manda* orbits on which they move), in the same way it does for these [interior] planets also.

The above passage exhibits the clear understanding Nilakantha had of the motion of the interior planets and the clinching argument which led him to this picture. This understanding that the interior planets go around the Sun, in the period that was given as the period of their *sighroccha* in the traditional planetary models, and that they go around the Earth in one year only because the Sun does so, has been very clearly set forth in the above passage. And this was the missing link, which had made all the earlier descriptions of planetary motion inadequate.

It was generally known to the ancients that the exterior planets, Mars, Jupiter and Saturn, go around the Earth and they also go around the Sun in the same mean period, because their geocentric orbit was outside that

of the Sun. Nilakantha was the first savant in the history of astronomy to clearly derive from his computational scheme (and not from any speculative or cosmological argument) that the interior planets go around the Sun and the period of their motion around Sun is also the period of their latitudinal motion. The fact that the mean period of their motion in longitude around the Earth is the same as that of the Sun is also explained as being due to their being carried around the Earth by the Sun.

Nilakantha also wrote a tract called *Grahasphutanayane Vikshepavasana*, where he has set forth his latitude theory in detail. There he has given the qualitative nature of the orbits of the Sun, Moon and the five planets in a single verse, which may be cited here.[16]

इन्द्वादेः स्वस्वपातद्वयत उदगवागर्धशः क्रान्तिवृत्तात्
विक्षिप्ता मान्दकक्ष्याः कथितनिजलवैः सर्वदा तुल्यसंख्यैः।
तत्रेन्दोर्मान्दकक्ष्या ह्यपमवलयमध्यस्थकेन्द्राः कुजादे-
र्मान्दाः कक्ष्या भगोलस्थितदिनकरकक्ष्यास्थमध्यार्ककेन्द्राः॥

The *mandakakshyas* (*kakshyamandala, pratimandala, karnavritta*, and *mandavritta*) of the Moon and others (Sun and the planets) are deflected equally to the north and south of the ecliptic from the two nodes [of their orbits] by amounts given for each of them [separately] and which remain fixed always. The *mandakakshya* of the Moon is centered at the centre of the ecliptic. For Mars and other planets (*Kujadi*), the centre of their *manda-kakshyas* [the centre of their *manda* deferent circles], is the mean Sun (*madhyarka*) which lies on the orbit of the Sun on the ecliptic.

Geometrical picture of planetary motion according to Nilakantha

Nilakantha presents a clear and succinct statement of the geometrical picture of planetary motion as implied by his revised planetary model in two of his small tracts, *Siddhantadarpana* and *Golasara*. We first present the version given in *Siddhantadarpana*:[17]

ग्रहभ्रमणवृत्तानि गच्छन्त्युच्चगतीन्यपि।
मन्दवृत्ते तदर्केन्द्वोर्घनभूमध्यनाभिकम्॥
मध्यार्कगति चान्येषां तन्मध्यं शीघ्रवृत्तगम्।

[16] *Grahasphutanayane Vikshepavasana* of Nilakantha, in *Ganitayuktayah*, Ed. K. V. Sarma, Hoshiarpur, 1979, p.63

[17] *Siddhantadarpana* of Nilakantha, Ed. K. V. Sarma, Hoshiarpur, 1976, p. 18.

तेषां शैघ्र्यं भचक्रात्र विक्षिप्तं गोलमध्यगम्।
शैघ्रत्वेन तदंशैः स्वं प्रमायोक्तं ज्ञशुक्रयोः।
मन्दवृत्तस्य चैवात्र क्षयवृद्धी स्वकर्णवत्।।

The [eccentric] orbits on which planets move (*graha-bhramana-vritta*) themselves move at the same rate as the apsides (*uccha-gati*) on *manda-vritta*, [or the *manda* epicycle drawn with its centre coinciding with the centre of the *manda* concentric]. In the case of the Sun and the Moon, the centre of the Earth is the centre of this *manda-vritta*.

For the others [namely the planets Mercury, Venus, Mars, Jupiter and Saturn] the centre of the *manda-vritta* moves at the same rate as the mean Sun (*madhyarkagati*) on the *sighra-vritta* [or the *sighra* epicycle drawn with its centre coinciding with the centre of the *sighra* concentric]. The *sighra-vritta* for these planets is not inclined with respect to the ecliptic and has the centre of the celestial sphere as its centre.

In the case of Mercury and Venus, the dimension of the *sighra-vritta* is taken to be that of the concentric and the dimensions [of the epicycles] mentioned are of their own orbits. The *manda-vritta* [and hence the *manda* epicycle of all the planets] undergoes increase and decrease in size in the same way as the *karna* [the hypotenuse or the distance of the planet from the centre of the *manda* concentric].

The version given in *Golasara* is very similar and is reproduced below:[18]

निजमन्दपरिधिगोच्चं केन्द्रीकृत्य भ्रमन्ति कक्ष्यासु।
विहगा रविचन्द्रमसोर्भगोलमध्यं स्वमन्दवृतिमध्यम्।।
अपमण्डलमध्यस्थस्वशीघ्रवृतिसंगतोच्चमन्येषाम्।
पाताद् विक्षिप्तमुदङ्मृदुवृत्तार्धं, ततोऽन्यतोऽन्यार्धम्।।
चन्द्रादीनां मन्दानुसारतः स्वस्वकक्ष्याः स्युः।
क्षयवृद्धी सर्वेषां परिधेर्मान्दस्य तु स्वकर्णवशात्।।

The geometrical picture described above is presented in Figs. 3,4. It is important to note that Nilakantha has a unified model for both the exterior and interior planets and the same is reflected in his formulation of the corresponding geometrical picture of planetary motion. Nilakantha's description of the geometrical picture of the planetary motions involves the notions of *manda-vritta* and *sighra-vritta*, which are nothing but the

[18] *Golasara* of Nilakantha, Ed. K.V. Sarma, Hoshiarpur, 1970, p. 11.

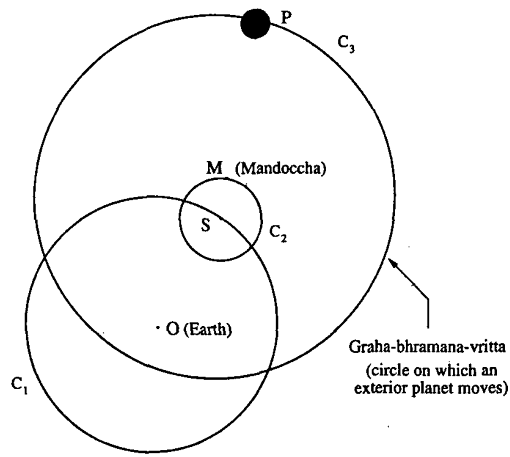

Fig. 3: Nilakantha's geometrical model of the motion of an exterior planet. The circles C_1, C_2 and C_3 represent *Sighra-vritta, Manda-vritta* and *Graha-bhramana-vritta* respectively. OS and SM represent *Sighrantya-phala* and *Mandantya-phala* respectively.

manda and *sighra* epicycles drawn with the centre of their concentric as the centre. These concepts are explained clearly in the beginning of the Eighth Chapter of the celebrated Malayalam treatise on mathematical astronomy, *Yuktibhasha*, of Jyesthadeva[19] who was a younger contemporary of Nilakantha.

An important point to be noted is that the geometrical picture of planetary motion as discussed above, deals with the orbit of each of the planets individually and does not put them together in a single geometrical model of the planetary system.[20] Each of the exterior planets have different *sighra-vrittas*, which are in the same plane as the ecliptic, and we have to take the point where the *adityasutra* (the line drawn from the centre along the mean Sun) touches each of these *sighra-vrittas* as the centre of their *manda- vritta.* On this *manda-vritta* the *mandoccha* is to be located, and

[19] *Yuktibhasha* of Jyesthadeva, Edited and Translated by K. V. Sarma (unpublished).

[20] The same is also true of the geometrical picture of planetary motion associated with the traditional planetary model as discussed by Paramesvara. This issue was not discussed in the reference cited in fn. 1.

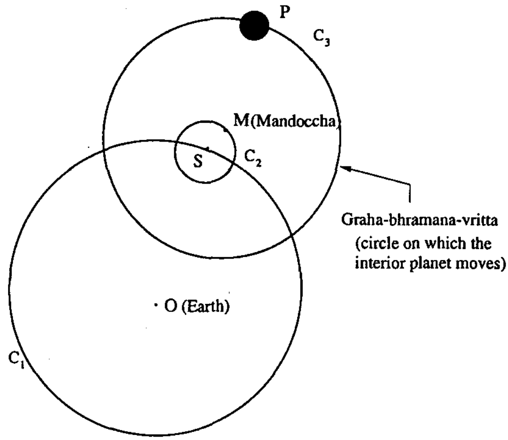

Fig. 4: Nilakantha's geometrical model of the motion of an interior planet. The circles C_1, C_2 and C_3 represent *Sighra-vritta, Manda-vritta* and *Graha-bhramana-vritta* respectively. OS and SM represent *Trijya* and *Mandantya-phala* respectively.

with that as the centre the *graha-bhramana-vritta* or the planetary orbit is drawn with the standard radius (*trijya* or Rsin 90). In the case of the interior planets, Nilakantha says that the *sighra-vritta* has to be drawn with the standard radius (*trijya* or Rsin 90) and the *graha-bhramana-vritta* is to be drawn with the given value of the *sighra* epicycles as the radii. In this way, we see that the two interior planets can be represented in the same diagram, as the *sighravritta* is the same for both of them.

To integrate the diagrams for all the planets into a single diagram of the planetary system, we shall have to use the notion of *bhu-taragraha-vivara* or the Earth-planet distance. Nilakantha has discussed this extensively in his *Aryabhatiyabhashya* and has shown how the effects of the latitudinal motions of the planets should be taken into account in the computation of the Earth-planet distance. The final picture that we would obtain, by putting all planets together in a single diagram adopting a single scale, is essentially what Nilakantha has described as the qualitative picture of planetary motion, that we presented earlier: The five planets, Mercury, Venus, Mars, Jupiter, and Saturn move in eccentric orbits around the mean Sun, which goes around the Earth. The planetary orbits are tilted with respect to the orbit of the Sun or the ecliptic, and hence cause

the motion in latitude. Since it is well known that the basic scale of distances are fairly accurately represented in the Indian astronomical tradition, as the ratios of the radius of the *sighra* epicycle to the radius of the concentric (*trijya*) is very nearly the mean ratio of the Earth-Sun and the Earth-Planet distances (for exterior planets) or the inverse of it (for interior planets), the geometrical picture of planetary motion will also be fairly accurate in terms of the scales of distances.

Quasi - Keplerian Model of Suryasiddhanta

S. Madhavan

Introduction

The description of planetary motion formulated by the early Indian astronomers falls broadly into two categories. One is the method of extrapolation based on the positions of planets for a long period and rectification of the accumulated errors. The second is the theory of epicycles. This essentially involves the assumption that each planet moves in a circle, the centre of which moves in another circle. In the case of the Sun and the Moon this is quite simple. Their motion is in a *mandavritta* or the epicycle, the centre of which moves in a *kakshyavritta* or deferent, which has the Earth for its centre. For Mars, Mercury, Jupiter, Venus and Saturn which have direct and retrograde motions, the picture is more complex. Their motion has to be described using a *mandavritta* and a *sighravritta*. In general, the motion is described with the epicycle theory. Though the *Aryabhatiya* school uses variable epicycles for all planets except the Sun and the Moon, *Suryasiddhanta* employs variable epicycles even for them, and this makes the theory unique. This introduces partial ellipticity, though not in the Keplerian sense. This procedure can also be explained in terms of the planetary theory described in *Suryasiddhanta*.

Ellipticity of planetary orbit in Suryasiddhanta

In this section we show that the variable epicycle method given in *Suryasiddhanta* corresponds to ellipticity of the planetary orbit. We first consider the Sun and the Moon which need only the *manda* correction. According to the model given in *Suryasiddhanta,* the planet moves in its *mandavritta* the centre of which moves in *kakshyavritta*, which has the Earth for its centre. Thus the mean planet moves in the *kakshyavritta*, and with the mean planet or the *madhyama* as the centre the actual/true planet moves in *mandavritta*. Both the mean and true planets take the same time to complete one revolution of the circles they describe, and this duration corresponds to the anomalistic period of the planet. The planet starts from *mandoccha*, the apex of slow motion or apogee. In the

case of the Moon the *uccha* has a forward motion of 40° per year, but in the case of other planets its motion is not significant. The *mandoccha* of the Sun given in *Suryasiddhanta* is 77° 14', and by modern computation, the figure for 500 AD would be 77° 15'. Ptolemy gives the value as 66° 15'.

The procedure for *manda* correction is as follows. First we find the mean longitude of the Sun and subtract it from *mandoccha* or the longitude of apogee. The remainder is called *mandakendra* or the mean anomaly denoted by *m*. The mean planet moves uniformly in *kakshyavritta* and it has to be rectified by the *manda* correction to get the true anomaly[1]. By the epicycle theory, the correction required is *(a/R)* sin *m*, in radians, where a is the radius of the *mandavritta* and *R* is the radius of *kakshyavritta*. *Suryasiddhanta* gives the *mandaparidhi* or the circumference of the *mandavritta* at the end of odd and even quadrants, taking the circumference of the *kakshyavritta* to be 360°. The procedure is explained below.

The *kakshyavritta* can be divided into four quadrants starting from the apogee. Let the *mandaparidhis* at the end of odd and even quadrants be a_1, a_2, and let $c = |a_1 - a_2|$. Then the *mandaparidhi* corresponding to the mean anomaly m is $a_2 - c\ |\sin m|$ if $a_2 > a_1$, and $a_2 + c\ |\sin m|$ if $a_2 < a_1$.

In the case of the Sun ,

$$a_1 = 13^0\,40',\ a_2 = 14^0,\ \text{and}\ c = 14^0 - 13^0\,40' = 20'.$$

The true anomaly corresponding to the mean anomaly would be

$$v = m - \frac{(a_2 - c|\sin m|\ )}{360^0}\ (R\sin m)$$

$$= m - \frac{a_2}{360^0}\ (R\sin m) + \frac{c|\sin m|}{360^0}\ (R\sin m). \qquad (1)$$

Adding *mandoccha* to this, the longitude of the Sun can be calculated. It can be verified that the use of a variable epicycle for finding the *manda* correction gives better values than the model which uses a fixed epicycle i.e., $a_1 = a_2 = 14^0$, as given by Aryabhata and Brahmagupta.

Now we shall consider the nature of the orbit. The epicycle theory is equivaent to the theory of eccentric circle. In the eccentric formulation

[1] For details regarding the true anomaly, mean anomaly etc., the reader is referred to the article by M.S.Sriram in this volume.

the planet moves in a circle uniformly, the geometrical centre of which is at a distance equal to the radius of the *mandavritta* from the centre of *kakshyavritta.*

Taking E, the Earth as the origin, the apse line $A'A$ as the X-axis and the line through E perpendicular to the apse line as the Y - axis, the equation of orbit is

$$(x-a)^2 + y^2 = R^2, \tag{2}$$

where a and R are the radii of the *mandavritta* and *kakshyavritta* respectively. When the radius a of *mandavritta* is variable,

$$a = a_2 - c\left|\sin m\right| = a_2 - c\frac{|y|}{R}.$$

Then Eq. (2) becomes,

$$\left[x-(a_2-\frac{c|y|}{R})\right]^2 + y^2 = R^2,$$

or

$$(x-a_2)^2 + \frac{2c(x-a_2)|y|}{R} + y^2\left(1+\frac{c^2}{R^2}\right) = R \tag{3}$$

Shifting the origin to $(a_2, 0)$, Eq.(3) becomes

$$x^2 + \frac{2cx|y|}{R} + y^2\left(1+\frac{c^2}{R^2}\right) = R^2.$$

Since,

(coefficient of x^2) (coefficient of y^2) $-$ (half the coefficient of xy)^2

$$= 1.\left(1+\frac{c^2}{R^2}\right) - \frac{c^2}{R^2} = 1 > 0,$$

the above equation represents an ellipse. Thus we get the two ellipses,

$$x^2 + \frac{2cxy}{R} + y^2\left(1+\frac{c^2}{R^2}\right) = R^2, \tag{4}$$

and

$$x^2 - \frac{2cxy}{R} + y^2\left(1+\frac{c^2}{R^2}\right) = R^2, \tag{4'}$$

in the upper and the lower half-planes, which together represent the path

of the planet. The xy term in Eq.(4) can be removed by rotating the axes of coordinates by

$$\theta = \frac{1}{2}\tan^{-1}\frac{\frac{2c}{R}}{\frac{-c^2}{R^2}} = \frac{1}{2}\tan^{-1}\left(-\frac{2R}{c}\right).$$

Then Eq.(4) takes the form

$$Ax^2 + By^2 = C. \tag{5}$$

Similarly, the xy term in Eq.(4') can be removed by rotating the axes by

$$\frac{1}{2}\tan^{-1}\left(-\frac{2R}{c}\right) = -\theta,$$

and then Eq.(4') takes the form

$$A'x^2 + B'y^2 = C. \tag{5'}$$

We observe that if the axes are rotated by an angle θ and xy term is removed, then the expression $ax^2 + 2hxy + by^2$ becomes $Ax^2 + By^2$, where,

$$A = a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta,$$

and

$$B = a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta.$$

By changing h to $-h$, Eq.(4) transforms to Eq.(4'), and by changing θ to $-\theta$, the effect of the latter rotation is obtained. Thus A and B remain the same when h is changed to $-h$, and θ is changed to $-\theta$. Therefore Eq.(5) and Eq.(5') represent the same ellipse. In other words, Eq.(4) and Eq.(4') represent the arcs of the same ellipse when it is rotated by angle θ and $-\theta$ about the major axis respectively. These two axes are joined at the points of intersection of Eq.(4) and Eq.(4'). To solve them, subtracting Eq.(4) from Eq.(4') gives $xy = 0$. Thus $x = 0$, or $y = 0$. Since Eq.(4) is for the upper half plane and Eq.(4') is for the lower half-plane, $x = 0$, y is positive for Eq.(4) and $x = 0$, y negative for Eq.(4'), and there is no common point. When $y = 0$, $x^2 = R^2$ *or* $x = \pm R$. Thus the common points are $(R,0)$ and $(-R,0)$. These are shown in Fig.(1). The curve is symmetric about the X axis, but not about the Y axis.

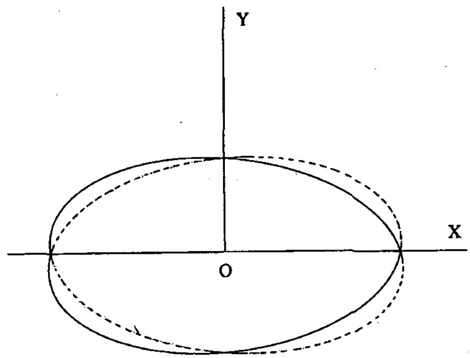

Fig. 1: The orbit of the Sun/Moon according to Suryasiddhanta
indicated by the continuous line.

We shall now reconcile this with the theory of planetary motion as given in chapter 2 of *Suryasiddhanta:*

अदृश्यरूपा: कालस्य मूर्तयो भगणाश्रिता:।
शीघ्रमन्दोच्चपाताख्या ग्रहाणां गतिहेतव:।।
तद्वातरश्मिभिर्बद्धास्तै: सव्येतरपाणिभि:।
प्राक्पश्चादपकृष्यन्ते यथासन्नं स्वदिङ्मुखम्।।
प्रवहाख्यो मरुत् तांस्तु स्वोच्चाभिमुखमीरयेत्।
पूर्वापरापकृष्टास्ते गतिं यान्ति पृथग्विधाम्।।

Forms of time, invisible in shape, situated in the zodiac called *sighroccha, mandoccha* and *pata* are the causes of motion.

The planets attached to these by cords of air are drawn towards and away by them with right and left hands forward and backward.

A wind called *pravaha* impels them towards their own *uccha;* as a result they (planets) get attracted eastwards or westwards and their motion is altered.

The essential idea here is that the motion of planets is due to the three forces at *mandoccha, sighroccha* and *pata.* The force at *pata* explains the changes in latitude. The forces associated with *mandoccha,* and *sighroccha* explain the direct and retrograde motion of planets. In the case of the Sun and the Moon, *mandoccha* force explains the motion. There is no change in the latitude of Sun and change in the latitude of Moon is explained through the force associated with *pata.*

We shall consider the force associated with *mandoccha* in the case of the Sun and the Moon. The motion in an eccentric circle is due to an attractive force towards the Earth. The change in the dimensions of the epicycle can be interpreted as an effect of a simple harmonic motion. We have

$$s = a_2 - c \sin m, \quad \text{where } 0 \leq m \leq 180^0 .$$

Differentiating with respect to the time t,

$$\frac{d^2 s}{dt^2} = ck^2 \sin m = ck^2 \left[\frac{-(s-a_2)}{c} \right] = -ck^2 \left[\frac{(s-a_2)}{c} \right]$$

where $\frac{dm}{dt} = k$, a constant. Thus

$$\frac{d^2 (s-a_2)}{dt^2} = -k^2 (s-a_2) .$$

Also,

$$s = a_2 + c \sin m, \quad \text{where } 180^0 \leq m \leq 360^0 .$$

We have,

$$\frac{d^2 s}{dt^2} = -ck^2 \left[\frac{(s-a_2)}{c} \right] = -k^2 (s-a_2) ,$$

and hence

$$\frac{d^2 (s-a_2)}{dt^2} = -k^2 (s-a_2) .$$

Thus the geometrical centre of the eccentric circle performs simple harmonic motion. The forces acting on the planets are:

(i) the centripetal force towards O, the geometrical centre of the circle, and

(ii) the force causing the simple harmonic motion of the geometrical centre.

The form of sighravritta

When it comes to the planets Mercury, Venus, Mars, Jupiter and Saturn, things are quite different. According to *Suryasiddhanta,* the mean position of the planet has to be obtained first. Then the mean position has to be

corrected using the *manda* and *sighra* corrections. *Mandoccha* or the apogee is given for each planet in the text. Except for the Moon, the motion of *mandoccha* is not significant. The number of revolutions of *sighroccha* in a *Mahayuga* is given for Mercury and Venus. For others *viz.,* Mars, Jupiter and Saturn, the *sighroccha* is the mean Sun. In the case of Mercury and Venus the mean position is the same as the mean Sun. The procedure outlined in *Suryasiddhanta* is the following:

> Find the mean position of the planet first. Apply half the *sighra* correction. From this position, calculate the *manda* correction and apply half of that. From this find the *manda* correction once again and apply it to the original mean position. Then calculate the *sighra* correction and apply it to the corrected mean position in the previous step.

This procedure for finding the planetary position is not followed by later writers. One can explain the rationale of *manda* and *sighra* corrections diagrammatically. What is important is that they should tally with observations. But what is the significance of half the correction? It may be a computational device.

Manda correction has been explained earlier. For *sighra* correction, even according to *Suryasiddhanta,* the mean position is the centre of *sighravritta*, in the third stage. The circumference of *kakshyavritta* is 360° and those of the *sighravrittas* are also given accordingly.

Let m= Corrected mean longitude − *Sighroccha* = *Sighrakendra.*

If a_1, a_2 are the *sighraparidhis* at the end of odd and even quadrants, and $c = |a_1 - a_2|$, then

sighraparidhi at the instant = $a_2 \pm c|\sin m|$,

as the case may be.

$$Dohphala = \frac{a_2 \pm c|\sin m|}{360^0} \times R \sin m.$$

One can find

$$Kotiphala = \frac{a_2 \pm c|\sin m|}{360^0} \times R \cos m.$$

From this one can get

$$Sphutakoti = R + Kotiphala.$$

One can write

$$Sighrakarna = \sqrt{a^2 + r^2 + 2Rr\cos m}\ ,$$

where R is the radius of the *kakshyavritta* = *trijya,* and r = radius of the *sighravritta.*

$$Sighraphala = \frac{Dohphala}{Sighrakarna} \times trijya.$$

It is necessary to examine the *sighravritta*. Let a be the radius of the *sighravritta,* when it is maximum and let b be the difference between the maximum and minimum values of the radius. The polar equation of *sighravritta* is

$$r = a + b\sin\theta,$$

where $0 \leq \theta \leq 180^0$. Then

$$\frac{1}{r} = \frac{1}{a + b\sin\theta} = \frac{a - b\sin\theta}{a^2 - b^2\sin^2\theta}$$

$$= \frac{1}{a^2}\left[a - b\sin\theta\right]\left[1 - \frac{b^2}{a^2}\sin^2\theta\right]^{-1}.$$

Since $\frac{b}{a}$ is small, we get,

$$\frac{a}{r} = 1 - \frac{b}{a}\sin\theta, \text{ approximately.}$$

This is an ellipse with $\frac{b}{a}$ as eccentricity. When $180^0 \leq \theta \leq 360^0$,

$$\frac{a}{r} = 1 + \frac{b}{a}\sin\theta$$

Thus we get two elliptical arcs, with common points at $\theta = 0$ *and* $\theta = 180^0$. Thus *sighravritta* can be approximately regarded as a curve obtained by joining two elliptical arcs. This is symmetrical about the line joining the common points, but is not an ellipse. These are shown in Fig.2.

Conclusion

We can conclude that *Suryasiddhanta* envisages curves formed by two elliptical arcs in the case of the *manda*-corrected orbit of the planet. The

sighravritta is also approximately so. From the computational point of view, perhaps some improvement is achieved by the variable epicycles. But the asymmetry it produces probably made the later astronomers deter from this.

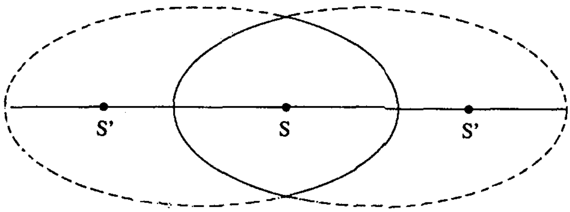

Fig. 2: Approximate form of *Sighravritta* is indicated by thick lines.
S is the common focus of the two ellipses and S' are the other two foci.

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True Longitudes of Planets and Variable Epicycles in the Aryabhatan School

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Introduction

In the Indian astronomical tradition, the practice of introducing *bijas* (corrections) to the parameters has been in vogue for long. The Indian astronomers were aware that the values of the governing parameters given by them, would be valid only for a limited period of time and that future competent astronomers should provide further improvements. For example, the celebrated Kerala astronomer, Paramesvara states:

> कालान्तरे तु सस्कार: चिन्त्यता गणकोत्तमै:।
>
> In course of time, the (necessary) corrections must be decided by the expert mathematicians.

In fact, Paramesvara in his extensive work on computations of eclipses - *Grahana-mandana,* observes in all humility that the times of contact etc., of an eclipse as given by him may at times differ slightly from observed positions कालेनानेन च सिद्ध: कदाचिदापि भिद्यते स्वल्पम्).

In the famous *Karana* text, *Laghumanasam,* Manjula (or Munjala) composed five *slokas* separately in the *arya* metre while the main text of 60 *slokas* was in *anustubh* metre. In the *arya* verses Manjula has given the planetary positions for his epoch. Giving reasons for the separate treatment of the five verses, the commentator Suryadeva Yajvan suggests:

> The epochal positions stated in those (five) verses will not serve for more than 100 years, and after every century thereafter, these will have to be replaced by new verses giving new epochal positions.

Again, the famous astronomer Jyesthadeva (c. 16th century) in his Malayalam text *Drikkarana* described the long series of revisions introduced over centuries in the Aryabhatan school of astronomy. He says:

(i) In the *Kali* year *"giritunga"* (i.e., 3623 ≡ 522 A.D), his work (*Aryabhatiyam*) was composed...... He had adjusted the (planets) revolutions by

reduction and addition in such a way that there was no zero correction at the beginning of *Kali.*

(ii) In course of time, deviations were observed in (the results arrived at by) this computation. Then, in the *Kali* year *mandasthala* (3785 ≡ 684 AD), several astronomers gathered together and devised, through observation, a system wherein (the correct mean longitudes were to be found) by multiplying the current *Kali* year minus *"giritunga"* (*Kali* 3623 i.e., the epoch). This system was named *Parahita* and many followed it, assuring themselves of its accuracy.

(iii) When a long time had elapsed, there occurred substantial deviations. Then a noble *Brahmana,* Paramesvara residing on the coast of the western ocean, revised the *Parahita* system by means of astronomical observations in the *Kali* year *"rangasobhanu"* (4532 ≡ 1431 AD).

(iv) The work *Tantrasangraha* by Nilakantha (Somayaji), with revised constants, is for twelve years later.

(v) The revolutions given therein (i.e., in the *Tantrasangraha*) too, becoming imperfect (in course of time), observations were continued by the astronomers on the west coast for 30 years, from the *Kali* year *"jaustava"* (4678 ≡1577 AD) through the *Kali* year *"jhanaseva nu"* (4709 ≡ 1607 AD), and by observation, the astronomical tradition was revised accurately.

(vi) Henceforth too, deviations between calculated and observed positions of planets should be carefully observed and revisions effected.

Again, there is a detailed statement in the *Brihat-tithicintamani,* by Ganesa Daivajna (16th cent. AD) describing how *sastra* which is *tathya* (accurate) at one period of time becomes *slatha* (inaccurate) and needs *samsthapana* (reestablishment) in any later period.

The celebrated astronomer Nilakantha Somayaji, referring to a certain commentator of Manjula's *"Manasam"* who laments, "Alas! we have precipitated into a calamity" (महति संकटे पतिता: स्म:), points out:

> ... One has to realize that the five *siddhantas* had been correct at a particular period. Therefore, one should search for a (new) *siddhanta* that does not show discord with actual observation (at the present time). Such accordance with observation has to be ascertained by (astronomical) observers during times of eclipses etc.

Bijas for civil days and revolutions, mandocchas, epicycles etc. of planets

In Indian astronomy, computations of true positions of planets are based mainly on the following parameters:

(i) Number of civil days in a *Mahayaga* (or *Kalpa*);

(ii) Mean rate of daily motion given in terms of the number of revolutions in a *Mahayuga* (or *Kalpa*);
(iii) The rates of motion of the *mandocchas* (apogees) of the planets in terms of revolutions in a *Mahayuga* (or *Kalpa*);
(iv) The peripheries of the *manda* and *sighra* epicycles of planets; and
(v) The epochal positions of bodies and special points.

In what follows, we propose suitable *bijas* (corrections) to the above parameters based on current values of various astronomical quantities. The parameters related to the items mentioned above are now considered one by one.

Civil days in a Mahayuga

In a *Mahayuga*, the Sun completes 432×10^4 revolutions; the period of one revolution with reference to fixed stars being defined as a sidereal solar year.

Taking the modern value of the sidereal Sun's daily motion as $SDM = 3548.1928098''$, the duration of a sidereal solar year becomes 365.2563627378105. However, allowing a maximum error of ± 5 in the eighth digit in the value of SDM, correspondingly, the duration of a sidereal solar year lies between 365.2563627429576 days and 365.2563627326635 days.

Accordingly, the number of civil days in a *Mahayuga* of 432×10^4 *years turns out to be 1577907487 days (ignoring the decimal part).* However, if the longer period of a *Kalpa* of 432×10^7 years is considered, we can have a more accurate figure for the number of civil days as 1577907487027.

Now, the numbers of civil days in a *Mahayuga* according to some important traditional texts and the proposed value are compared in Table 1.

Table 1: Civil days in a *Mahayuga*

S.No.	Text	No. of civil days
1.	Aryabhata I (*Aryabhatiya*)	1,57,79,17,500
2.	Brahmagupta's *Khandakhadyaka*	1,57,79,17,800
3.	*Suryasiddhanta* (SS)	1,57,79,17,542
4.	Aryabhata II (*Mahasiddhanta*)	1,57,79,17,542
5.	Bhaskara II (*Siddhantasiromani*)	1,57,79,16,450
6.	Proposed modern value	1,57,79,07,487

Proposed *bija* of SS value is -10,341. It may be noted that Bhaskara II (b.1114 AD) suggested a *bija* of -1,378 to the SS value.

Revolutions of bodies in a Mahayuga

The mean motion of a heavenly body is determined from its number of revolutions in a *Mahayuga* of 432×10^4 years.

In our proposed values for the revolutions of the Sun, the Moon and other bodies, we have considered the rate of daily change in the sidereal longitude of these bodies. In Table 2, the numbers of revolutions of the bodies according to the traditional texts are compared with the suggested modern values.

Peripheries of manda epicycles

The expression for the *manda* equation is given by

$$E = \frac{a}{R} \sin m, \tag{1}$$

Table 2: Revolutions of celestial bodies in a *Mahayuga*

S. No.	Celestial body	Aryabhata I	Brahma-gupta	*Surya-siddhanta*	Bhaskara II	Proposed Modern
1.	*Ravi*	43,20,000	43,20,000	43,20,000	43,20,000	43,20,000
2.	*Chandra*	5,77,53,336	5,77,53,336	5,77,53,336	5,77,53,300	5,77,52,986
3.	*Chandra's Mandoccha*	4,88,219	4,88,219	4,88,203	4,88,206	4,88,125
4.	*Rahu*	2,32,226	2,32,226	2,32,238	2,32,311	2,32,269
5.	*Kuja*	22,96,824	22,96,824	22,96,832	22,96,828	22,96,876
6.	*Budha's Sighroccha*	1,79,37,020	1,79,37,000	1,79,37,060	1,79,36,999	1,79,37,034
7.	*Guru*	3,64,224	3,64,220	3,64,220	3,64,226	3,64,195
8.	*Sukra's Sighroccha*	70,22,388	70,22,388	70,22,376	70,22,389	70,22,260
9.	*Sani*	1,46,564	1,46,564	1,46,568	1,46,567	1,46,656

Note: The traditional *siddhantas* have given the revolutions of the bodies in a *Kalpa* of 432×10^7 years. However, in Table 2, we have reduced them for a *Mahayuga* of 432×10^4 years (by dividing the figures by 1000), and rounded off to the nearest integer.

where a is the periphery (in degrees) of the *manda* epicycle, $R = 360^0$ and m is the *manda* anomaly of the body. The corresponding modern formula for the equation of centre, considering the first two terms is

$$E=\left[2e-\frac{1}{4}e^3\right]\sin m+\left[\frac{5}{4}e^2-\frac{11}{24}e^4\right]\sin 2m, \tag{2}$$

where e is the eccentricity of the body's elliptical orbit. Generally, since e is small, ignoring the higher powers of e, the equation of centre is approximated as

$$E = (2e)\sin m. \tag{3}$$

However, in proposing *bija* to the peripheries of the *manda* epicycles of the heavenly bodies, we now consider even the higher powers of e viz., e^2, e^3 and e^4.

Let $e_1 = 2e - \frac{1}{4}e^3$, and $e_2 = \frac{5}{4}e^2 - \frac{11}{24}e^4$

Then (2) can be written as

$$E = e_1 \sin m + e_2 \sin 2m$$
$$= e_1 \sin m + 2e_2 \sin m \cos m$$
$$= (e_1 + 2e_2 \cos m)\sin m. \tag{4}$$

Table 3 : Peripheries of *manda* epicycles

Celestial body	Periphery (Deg) MIN	Periphery (Deg) MAX	Coefft. of equation of centre MIN D	M	S	MAX D	M	S
Ravi	11.80781	12.31284	1	52	45	1	57	34
Chandra	36.80850	42.22932	5	51	29	6	43	15
Budha	109.80020	184.71950	17	28	30	29	23	56
Sukra	4.86900	4.952738	0	46	29	0	47	17
Kuja	59.30061	74.92371	9	26	16	11	55	28
Guru	32.69004	36.89168	5	12	10	5	52	17
Sani	37.41842	43.03509	5	57	19	6	50	57
Uranus	31.42751	35.29051	5	0	6	5	36	59
Neptune	6.40495	6.55065	1	1	9	1	2	33
Pluto	123.00710	230.98760	19	34	37	36	45	46

Here, it is interesting to note that the coefficient of sin m is a variable and that most of the traditional Indian texts indeed have taken the coefficient of sin m viz., a/R as a variable. In Table 3, the proposed

peripheries (in degrees) for the different celestial bodies based on (4) above are given, using the current values of the eccentricities of the orbits of the bodies.

The second and third columns give respectively the minimum and the maximum values of the peripheries (in degrees) of the *manda* epicycles.

Correspondingly, the third and the fourth columns provide respectively the minimum and the maximum values of the coefficients (in deg., min and sec.) of sin *m* of the *manda* equation (1). We have included Uranus, Neptune and Pluto also in the list of planets.

The eccentricity e of the Earth's orbit is given by

$$e = 0.01675104-(0.0000418)\,T-(1.26 \times 10^{-7})\,T^2,$$

where T is the number of Julian centuries (of 36525 days each) completed since the epoch 1900 AD. Accordingly, over centuries, the eccentricity e and hence the periphery of the *manda* epicycle change.

Peripheries of sighra epicycles of planets

The peripheries of the *sighra* epicycles of the different planets given in various *siddhantic* texts are presented in Table 4. In fact, it is found that the periphery of the *sighra* epicycle of a planet is related to its mean heliocentric distance a. For the interior planets *viz., Budha* and *Sukra*, the periphery of the *sighra* epicycle is given by $p = 360^0 a$, where a is in astronomical units. One astronomical unit (a.u.) is defined as the mean distance of the Earth from the Sun.

In Table 4, the peripheries of the *sighra* epicycles of the planets according to the traditional texts are compared with the modern values (considering the mean heliocentric distances of the planets). In the case of the exterior planets *viz., Kuja, Guru, Sani* and the trans-Saturnine planets, the periphery is given by $p = \dfrac{360^0}{a}$.

According to modern astronomy, the semi-major axis a of the elliptic orbit as well as its eccentricity e of a planet gradually change with time. The semi-minor axis, b of the orbit is given by $b = a\sqrt{1-e^2}$, at any given time.

The actual distance (radius-vector) of a planet from the Sun varies between the semi-minor axis b and the semi-major axis a, for small e. Accordingly, it is conjectured that the periphery of the *sighra* epicycle

varies (i) from $p = \frac{360^0}{a}$ to $p = \frac{360^0}{b}$ for an exterior planet, and (ii) from $360^0 b$ to $360^0 a$, for an interior planet.

Table 4: Peripheries of *sighra* epicycles (in degrees)

Planet	Mean heliocentric distance a	*Aryabhatiya*	*Khanda-khadyaka*	*Surya-siddhanta*	Modern
Budha	0.387099	130.5-139.5	132	132-133	139.3555
Sukra	0.723332	256.5-265.5	260	260-262	260.3994
Kuja	1.523692	229.5-239.5	234	232-235	236.2682
Guru	5.202561	67.5-72	72	70-72	69.1967
Sani	9.554747	36.5-40	40	39-40	37.6776

Moon's equations

In computing the position of the Moon, according to the *siddhantic* texts, there has always been a noticeable deviation. The ancient Indian astronomers suggested the well known corrections, besides the *manda* equation, namely the evection and variation.

The equation of centre (*manda* correction) was known in India even before Aryabhata I (476 AD). Aryabhata gives the coefficient in the *manda* equation as 300'.25. Brahmagupta in his *Uttara Khandakhadyaka* gives the same as 301'.7. However, it must be pointed out that, out of the actual equation of centre, a part is combined with the second correction ("evection") and the combined equation is given in later *siddhantic* texts.

In fact, this combined equation for the Moon was first given, among the Indian astronomers, by Manjula (or Munjala, 932 AD) in his *Laghumanasa.* P.C.Sengupta points out, "In form the equation is most perfect, it is far superior to Ptolemy's; it is above all praise". While the credit of discovering the Moon's second equation among the Hindu astronomers undoubtedly goes to Manjula, it was Bhaskara II (b. 1114 AD) who introduced it into his *siddhanta.*

The third equation for the Moon's position, "variation" was introduced in Indian astronomy by Bhaskara II in 1152 AD, four centuries before Tycho Brahe discovered it in the west.

The honour of introducing the fourth equation to the Moon's position, now called "annual equation" goes to the highly dedicated astute astronomer from Orissa, M.M. Samanta Chandrasekhara Simha of the

19th century. He called it *"Digamsa" samskara* and incorporated it in his remarkable text, *Siddhantadarpana*. The constant coefficient in Chandrashekhara's equation is 11'27".6. It is important to note that Tycho Brahe had given the coefficient as 4'30". The modern value is 11'10". Thus, Chandrashekhara Samanta's value is far closer to the modern value. This accuracy of his value is truly remarkable in the light of the fact that the Samanta was trained exclusively in the Sanskrit astronomical tradition and had no acquaintance with the western developments in astronomy.

The modern expressions for the three above-said equations of the Moon are as follows:

1. Evection $= 4586'' \sin\,(2D-g)$,

where $D = M - S$, the mean elongation of the Moon (from the Sun), M and S being the mean longitudes of the Moon and the Sun respectively and g is the mean anomaly of the Moon (from its perigee).

In the context of Indian astronomy, the mean anomaly (*mandakendra*) is measured from the apogee (*mandoccha*). If the perigee and the apogee of the Moon are denoted respectively by P and A, then we have

$$\text{Mean anomaly, } g = M - P = M - (A + 180^0)$$
$$= (M - A) - 180^0,$$

so that the evection equation becomes

$$\text{Evection} = 4586'' \sin\left[2D - \left\{M - (A + 180^0)\right\}\right]$$
$$= -4586'' \sin\left[2D - (M - A)\right].$$

However, as defined in the *Suryasiddhanta*,

Manda anomaly = *Mandoccha* - Mean longitude $= A - M$,

in which case,

$$\text{Evection} = -4586'' \sin(2D + MA),$$

where $MA = A - M$, the *manda* anomaly of the Moon. In terms of the mean longitude of the Sun (S) and the Moon (M) and the *mandoccha* A, we have,

$$\text{Evection} = -4586'' \sin\,(M - 2S + A).$$

2. Variation $= 2370'' \sin\,(2D)$,

where $D = M - S$, the Moon's elongation from the Sun. Chandrashekhara Samanta in his *Siddhantadarpana* has taken this equation as $2292'' \sin(2D)$.

3. Annual equation $= -668'' \sin(g')$,

where g' is the Sun's mean anomaly $i.e., g' = S - P'$, where P' is Sun's perigree.

Here also, considering the Sun's anomaly as being measured from its apogee (*mandoccha*) A', as is the case in the *siddhantas*, we have

$$\text{Annual equation} = -668'' \sin\left[S-(A'+180^0)\right]$$

$$= -668'' \sin(A'-S),$$

where $(A'-S)$ is the Sun's *mandakendra* (as defined in the *Suryasiddhanta*).

Remark:

In fact, the Moon's annual equation happens to be a fraction of Sun's *manda-phala* (equation of centre). According to the modern values of the concerned coefficients, we have,

Sun's equation of centre $= 6910'' \sin(g')$

Annual equation of the Moon $= -668'' \sin(g')$

The ratio of the latter to the former $= \dfrac{-668}{6910} \approx \dfrac{-1}{10.34}$

Chandrashekhara Samanta has approximated this ratio to $\dfrac{-1}{10}$ and taken

Moon's annual equation $= \dfrac{-1}{10} \times$ Sun's *manda* equation.

As a respectful tribute to *Mahamahopadhyaya* Samanta Chandrasekhara Simha, we shall continue to use the names *Tungantara, Pakshika* and *Digamsa* given by him, respectively for evection, variation and the annual equation.

Thus, from (1), (2) and (3), we have the three equations of the Moon, besides the usual *mandaphala,* given by

i. Tungantara Samskara (Evection) $= -4586'' \sin(M-2S+A)$,

ii. Pakshika Samskara (Variation) $= 2370'' \sin 2(M-S)$,

iii. Digamsa Samskara (Annual equation)

$$= -668' \sin(MK) \approx \frac{-1}{10.34} \times (\text{Sun's equation}),$$

where MK is the Sun's *mandakendra* $= A'-S$, where A' is Sun's *mandoccha.*

Note: In the *Tungantara* equation, as given by Chandrasekhara Samanta, a part of the (modern) equation of centre is combined with the evection equation. In the earlier *siddhantic* texts also, the *manda* equation included only a major part of the (modern) equation. However, in the proposed equations we are suggesting, the three equations (i) to (iii) above correspond to the three equations adopted in modern astronomy.

The case of Budha and Sukra

The traditional Indian astronomical texts have always treated *Budha* and *Sukra* differently from the remaining *taragrahas viz., Kuja, Guru* and *Sani* in the context of determining their true positions.

(i) While the mean positions of the exterior planets are taken as they are, in the case of *Budha* and *Sukra* (the interior planets), two special points called *Budha sighroccha* and *Sukra sighroccha* are considered.

The position of the mean *Ravi* is itself taken as the position of both mean *Budha* and mean *Sukra*. Again, while working out the *sighra* equation, the argument of the relevant sine function is taken, for example in the case of *Budha*, as $(B-R)$ as per the *Suryasiddhanta* convention, where B and R are respectively the *Budha sighroccha* and the mean *Ravi*. In the case of the exterior planets, the order of the terms in the argument is reversed. For example, for *Guru*, the argument in the sine term of the *sighra* equation is $(R-G)$ where G is the mean position of *Guru*. For the exterior planets, the mean *Ravi* is considered as their *sighroccha*.

Nilakantha Somayaji (1444 - 1545 AD) points out in his *Tantra-sangraha* and *Aryabhatiyabhashya* that it is incorrect to have such a different treatment of the interior and exterior planets and that the Sun is the common centre for the *sighra* equation to all the planets. This is truly a remarkable breakthrough in the history of mathematical astronomy in general and in Indian astronomy in particular (see K. Ramasubramanian *et al.*, Current Science, May 1994). In fact, Nilakantha's innovation, prompted by his *Paramaguru* Paramesvara (1380 - 1460 AD), is highly suggestive of a heliocentric model of planetary motion, much before Copernicus.

Following the innovation introduced by the celebrated Kerala astronomer Nilakantha, we propose the consideration of the *sighra* anomaly in the same way for all the planets, namely

Sighrakendra = *Sighroccha* - Mean planet

(in the *Suryasiddhanta* style), where the *sighroccha* is the mean Sun for all the planets.

(ii) In the case of *Budha* (Mercury), the major reason why its calculated true position, according to *siddhantas* goes generally off the mark is that the *manda* periphery taken for *Budha sighroccha* by the *siddhantas* is far below its actual expected value.

The *manda* pheriphery, as prescribed in the *Suryasiddhanta*, varies from 28° to 30° for *Budha*. But actually, as pointed out in Table 3, the periphery of *Budha*'s *manda* epicycle should vary from about 109°.8 to about 184°.7. The coefficient of the *manda* equation depends on the eccentricity, e of the planet's orbit. In fact, the coefficient, to a first approximation is given by $2e$. In the case of *Budha*, $e = 0.205656$, currently, and hence the coefficient of the *manda* equation turns out to be about 23°.566. In our proposed improved values for the parameters (*vide* Table 3), we have considered even higher powers of e upto e^5 and the first two terms in the expansion for the equation of centre. Accordingly, for the beginning of 2000 AD, the variation of the periphery of the *manda* epicycle is from 109°50'46" to 184°47'43", depending on the varying *manda* anomaly. Correspondingly, the coefficient of the *manda* equation varies from about 17°.48 to 29°.41. In fact, the ancient Indian astronomers had not taken note of the fact that among the planets known to them, *Budha*'s orbit has the largest eccentricity. This lapse explains why *Budha*'s true position always remained an enigma to them, despite all the *bija samskaras* introduced by them. Of course, now after the discovery of the three trans-saturnine planets, Pluto's orbit has the largest eccentricity (0.2488723), while the immediately next is that of Mercury (0.205656).

Mean positions of bodies at Kali epoch

The mean positions of the Sun, the Moon (with its *mandoccha*, anomaly from the apogee and the ascending node, *Rahu*), and all the planets, upto Pluto, at the *Kali* epoch (Mean midnight between 17/18 February 3102 B.C. at Ujjain) are computed according to modern formulae. The *Sayana* (tropical) as well as the *Nirayana* (sidereal) longitudes are listed in Table 5. The mean *Ayanamsa* adopted for the *Kali* epoch is -46°34'52" based on the recommendations of the Calendar Reform Committee of the Govt. of India.

Date (Julian)	:	18 Feb. 3102 BC
Time (IST)	:	0 Hours 27 Minutes (local midnight)
Place	:	Ujjain

Longitude	:	75° 46'E
Latitude	:	23°11'N
Julian Days	:	588466, Week day : FRIDAY

Table 5 : Mean positions of celestial bodies at *Kali* epoch
(*Ayanamsa* : 46°34'52)

Celestial Body		*Sayana*			*Nirayana*	
	D	M	S	D	M	S
Chandra	297	41	10	344	16	2
Rahu	142	12	36	188	47	28
Ravi	301	37	54	348	12	46
Sukra	333	52	43	20	27	35
Budha	267	27	27	314	2	19
Kuja	289	2	33	335	37	25
Guru	26	41	57	73	16	49
Sani	349	49	59	36	24	51
Uranus	344	50	57	31	25	49
Neptune	250	1	11	296	36	3
Pluto	317	21	11	3	56	3

Table 6 : *Mandocchas* of celestial bodies at *Kali* epoch
(*Ayanamsa* : 46°34'52)

Mandoccha of the celestial body		*Sayana*			*Nirayana*	
	D	M	S	D	M	S
Chandra	66	31	36	113	6	28
Ravi	15	59	32	62	34	24
Kuja	109	2	33	155	37	25
Budha	178	5	2	224	39	54
Guru	191	54	28	238	29	20
Sukra	293	54	36	340	29	28
Sani	325	46	24	12	21	16
Uranus	349	2	35	35	37	27
Neptune	70	1	11	116	36	3
Pluto	333	0	18	19	35	10

Note : In Tables 5 and 6, (i) Apogee is the *mandoccha* and the anomaly is the *mandakendra* measured from the apogee; (ii) in the case of *Budha* and *Sukra,* the mean positions of the planets themselves are considered and not their *sighroccha.* As pointed out earlier, the mean Sun is now the *sighroccha* for all the planets; (iii) the *Suryasiddhanta* has taken the Moon's *mandoccha* at the

Kali beginning as 90°, while our modern computation for the (*Nirayana*) *mandoccha* brings it to 113° 6' 28".

Revolutions of bodies in a *Kalpa*

Table 7 : Revolutions of celestial bodies in a *Kalpa*

Celestial Body	Revolutions of the body	Revolutions of the *Mandoccha*
Chandra	57,75,29,85,910	48,81,25,074
Ravi	4,32,00,00,000	38,777
Budha	17,93,70,33,867	19,134
Sukra	7,02,22,60,402	1,439
Kuja	2,29,68,76,453	53,367
Guru	36,41,95,066	25,671
Sani	14,66,56,219	67,486
Uranus	5,14,16,997	10,607
Neptune	2,62,19,242	3,424
Pluto	1,73,90,083	214
Rahu	23,22,68,618	

In Table 2, we had provided the numbers of revolutions (rounded off to the nearest integer) executed by the heavenly bodies and the special points in the course of a *Mahayuga* of 432×10^4 years. However, to get more accurate values for the mean positions of the bodies we have listed the numbers of revolutions in a *Kalpa* $(432\times10^7\ \mathit{years})$, based on the modern known rates of (sidereal) motion of these bodies in Table 7. In this table the revolutions of the (sidereal) *mandocchas* of all the planets, including Uranus, Neptune and Pluto also are provided.

Conclusion

In the present paper we considered the significance of variable peripheries of the *manda* epicycles of the planets in obtaining their true celestial longitudes. We have also suggested *bijas* (corrections) for updating the various parameters in obtaining the true positions of planets, comparable to modern results.

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Samanta Chandra Sekhara and His Treatise Siddhantadarpana

L. Satpathy

Introduction

Astronomy has been rightly called the mother of all sciences. It is the oldest branch of science which gave birth to the laws of motion in dynamics, paving the founda-tion for the emergence of modern science through the successive works of Copernicus, Tycho Brahe, Kepler, Galileo and Newton. India has a rich heritage in astronomy right from the Vedic period. References to *nakshatras* and names of the months used in the calendar even today can be found in *Rigveda* which is the oldest of all the *Vedas,* and also believed to be the oldest literature in the world. Researches in this area has continued in India all along. The galaxy of astronomers born in India includes geniuses like Aryabhata, Varahamihira, Brahmagupta and Bhaskara, who worked and enriched this field in a period spanning from fifth century AD to twelfth century AD. The tradition of *siddhantic* astronomy continued beyond twelfth century right up to the beginning of the twentieth century, finally coming up to Samanta Chandra Sekhara (1835-1904 AD) of Orissa and his treatise *Siddhantadarpana.*

Samanta Chandra Sekhara occupies a place of special importance in this tradition, being the last link in a long chain of astronomers. In every branch of science, the successive scientists have to improve upon the works of their predecessors through their own research. This culture, more or less universal, is inherent in the very nature of the evolution of any branch of knowledge. Naturally, Samanta had the onerous responsi-bility of improving upon the available knowledge. He fulfilled this role by identifying the errors accumulated over ages in the parameters used for the computation of longitudes etc., and also by enunciating new parameters and procedures. To achieve this, he fabricated ingenious instruments for precision astronomical observations, collected data of unprecedented accuracy, devised new methods of calculation and gave a new model of planetary system. All his work was carried out without the use of telescope or any other optical instruments whose existence he was

unaware of, and was in line with the work of his predecessors centuries before. It is no exaggeration to say that with his lifelong effort, *siddhantic* astronomy reached again the zenith of its glory, before being consigned to history. The International Journal Nature [1] in 1899, has not only compared Samanta with Tycho Brahe but eulogised him to be even greater than Tycho.

Life

Samanta Chandra Sekhara was born in a royal family of the princely estate Khand-para in Orissa in 1835. It is a small village situated about sixty miles west of Bhubaneswar, surrounded by hills and jungles. His father's name was Shyamabandhu and mother's name Bishnumali. He is also fondly called Pathani Samanta in Orissa. From the very childhood he showed extraordinary qualities. At the age of four, he spotted the planet Venus in the sky, in the day time. In those days, it was considered inauspicious to see a star in the sky during the day time. So his father had to perform an elaborate *Yagna* before Lord Jagannath to get rid of the sin. He received primary education from a Brahmin Sanskrit teacher. Thereafter, he started teaching himself *Lilavati, Bijaganita,* works on *Vyakarana* etc., and more importantly, the ancient astronomical works *Suryasiddhanta* and *Siddhantasiromani* from the family library.

At the age of fifteen, he began to check the predictions of the *siddhantas* with his astronomical observations. He was surprised to find that the predictions of the classics like *Siddhantasiromani* and *Suryasiddhanta* did not agree with his observations. The stars and planets, neither appeared at the right place in the sky, nor at right time, as per the calculations. Disagreement between repeated calculations and observations finally confirmed him that ancient *siddhantas* had accumulated errors and there was need for their reformulation to get results which are in agreement with observations. This kind of conviction against the time honoured age-old scientific treatises, on the part of a boy at the tender age of fifteen, is undoubtedly rare indeed.

Young Chandra Sekhara resolved to rectify all those errors accumulated in Indian astronomy over time. The major hurdles on his way were getting the right instruments for observation. Ancient Indian works did not give details of instruments or methods of measurements explicitly, except hints here and there. He devised his own instruments for the measurement of time, height of distant objects, latitude and longitude of heavenly bodies, etc. His passion for precision and accuracy

was unbelievable. He constructed as many as nine types of instruments for measuring time. His most versatile instrument is *Manayantra* which is very simple, yet very versatile. From the age of fifteen upto the age of twenty-three, he went on recording his observations and devising formulae for astronomical calculations. Two years later, he started recording his results in the form of *slokas* and wrote a treatise called *Siddhantadarpana*, which was completed in 1869 when he was thirty-four. Working in a remote corner of Orissa, far from Cuttack, the only town with some semblance of modern education then, he had no option but to keep the manuscript written on palm leaves in Oriya script for thirty years, lying in a corner of his house.

Prof. Mahesh Chandra Nyayaratna, Principal of Sanskrit College Calcutta, was in charge of Sanskrit education of Bengal Presidency, which comprised then the present Bengal, Bihar and Orissa provinces. In one of his official tours, he providentially met Samanta and was greatly impressed with his erudition and scholarship, and probably he introduced him to Prof. Jogesh Chandra Ray of Cuttack College, presently called Ravenshaw College. Later, it was the recommendation of Prof. Nyayaratna which brought him the title of *Mahamahopadhyaya* conferred by the British government in 1893.

Prof. Jogesh Chandra Ray played a key role in the publication of *Siddhantadarpana* in Devanagari script from a Calcutta press in 1899 with the financial support from the kings of Athmalik and Mayurbhanj. It must be noted that the scholarly introduction of fifty-six pages in English therein by Prof. Ray formed the window through which the outside world could get a glimpse of the valuable treasure contained in this monumental work in Sanskrit verses.

Though he belonged to a royal family, Samanta Chandra Sekhara had to face lot of hardships to maintain his large family consisting of six daughters, five sons and a large number of dependents. He had a Jaghir of two villages and a small amount of land out of which he had an annual income of Rs.500/-, and 1000 maunds of paddy. Six months before his death, the government granted him an allowance of Rs.50/- per month. The king Natabar Singh was extremely envious of him for his popularity and put all kinds of hurdles on his way.

Samanta was a very religious person. A large part of his daily life was devoted to prayer, worship and meditation. He breathed his last at Puri on the twelfth day of the dark fortnight of the month of *Jyestha* in 1904.

Siddhantadarpana

Samanta in the 17th *sloka* of the first chapter of the treatise enumerates the requisite characteristics of a *siddhanta.* The work that deals with the theoretical division of time from the smallest unit of *Truti* (0.274348×10^{-6} sec) to *Pralaya* (10^{13} solar years), along with the motion of the celestial objects, their rotations, orbits, alignments, occultations and eclipses etc., with the relevant mathematics like algebra, arithmetic, geometry and trigonometry and also concerns with the question of origin of the universe, is called a *siddhanta.* One will notice that *Siddhantadarpana* with 24 chapters and 2500 *slokas* out of which 2284 are Samanta's own composition and 216 are citations from the earlier authors, falls short of none of these qualifications. Samanta has made original contributions to most of the aforesaid topics, dealt in his astronomical treatise.

It will not be out of place here to give a brief contentwise description of *Siddhantadarpana.* It is broadly divided into two parts, i.e., *Purvardha* (first half) and *Uttarardha* (latter half). The first half contains fifteen chapters and the latter half, nine. The chapters are further grouped under five sections, namely, *Madhyamadhikara, Sphutadhikara, Triprasnadhikara, Goladhikara* and *Kaladhikara.* The first two sections deal with the mean motions and true positions of planets respectively. The third section deals with motion described in terms of space, time and direction. The fourth section gives an account of the relevant mathematics like spherical trigonometry, geometry etc. The fifth section describes different ways of reckoning time. The distribution of chapters and *slokas,* with the title of the topic dealt in each chapter is given in Table 1.

Table 1: Contents of *Siddhantadarpana*

Name of Section	Chapter Number	Chapter contents	Number of *slokas* in the Chapter
Madhyamadhikara	1	Description of time	55
	2	Planetary revolutions, etc.	25
	3	Mean planetary positions	77
	4	Various corrections	57
Sphutadhikara	5	True planet positions	221
	6	Finer corrections	161
Triprasnadhikara	7	Gnomons, etc.	94
	8	Lunar eclipse	87

	9	Solar eclipse	78
	10	*Parilekha* description	37
	11	Transit etc., of planets	111
	12	Alignments of planets	93
	13	Rising and setting of planets	84
	14	Phases of Moon	67
	15	Description of Mahapata	70
Goladhikara	16	A set of questions	79
	17	Description of Earth	159
	18	Description of Earth (contd.)	175
	19	The celestial sphere	123
	20	Description of instruments	111
	21	Some deeper questions	249
Kaladhikara	22	Description of years, etc.	76
	23	Prayer to *Purushottama*	55
	24	Conclusions	156
Total			2500

The contents of *Siddhantadarpana* look amazing as the achievement of a single author. Chandra Sekhara has observed, verified and corrected wherever necessary all that was known to the Hindu astronomers over time. Very often he has gone beyond them to discover new phenomena and to give new formulations.

Contributions

Samanta has made many signal contributions to Indian Astronomy. His main contributions can be seen in some detail in the elaborate introduction to *Siddhantadarpana* written by J.C. Ray [2]. A brief account can be seen in Refs. [3,4]. Here we will present only some important features of his work.

(a) Sidereal periods

Samanta has given the sidereal periods of all the planets; here we consider that of Saturn only to show the degree of accuracy [3] he could reach through his naked eye observations. He gave this value as 10759.7605 mean solar days. *Suryasiddhanta and Siddhantasiromani* had given the corresponding value as 10765.7730 and 10765.8152 mean solar days respectively. The European astronomy of 1899 and modern astronomy of 1994, using highly sensitive optical instruments have obtained the

values 10759.2197 and 10759.2300 mean solar days. It may be noted that while ancient values differ from the modern values by more than six days, Samanta's result is quite close in spite of the fact that his calculations were based on naked eye observations.

(b) Moon's motion

Samanta has discovered the important irregularities in the motion of the Moon called *Tungantara* (evection), *Pakshika* (variation) and *Digamsa* (annual equation) which are shifts from the expected mean position of Moon. His calculated maximum corrections are 2°40', 38'12" and 12' which may be compared with the modern values, 1°17', 39'31" and 11'9" respectively. It must be mentioned that he was the first Indian astronomer to recognise all the three irregularities. The earlier astronomers had only observed *Tungantara* and *Pakshika.* Tycho Brahe in Europe was the first western astronomer to have detected and measured all the three irregularities. Due to these discoveries, Samanta's calculation of solar and lunar eclipses became more accurate.

(c) Sun-Moon distance

The distances of the Sun and Moon from the Earth are important quantities that enter into the calculation of astronomical events. In ancient Indian astronomy, the ratio of these two distances is widely used. Samanta had worked for several years on this problem. Once, while he was entering the courtyard of his home, he noticed a beautiful image of the Sun on the wall, formed by the Sun rays passing through an aperture in the adjoining fence made of palm leaves. Measuring the height of the image of the Sun and the distance of the wall from the fence, he got an important clue to this problem. The ancient Indian astronomers placed the Sun at a distance of not more than 14 times that of the Moon from the Earth. Samanta hiked this ratio by more than 10 times to 154 which is much closer to the modern value 390. Due to this, the accuracy of his predictions of solar and lunar eclipses improved dramatically compared to his predecessors.

(d) Planetary model

The most prominent of Samanta's theoretical contribution is his model of planetary system, which is different from those of his predecessors in

India. His model supposes that the Earth is stationary, and that the Sun, Moon and other planets revolve around it. But he has the novel feature that planets Mercury, Venus, Mars, Jupiter and Saturn, all go around the Sun, and the Sun together with these companions moves around the Earth. Thus he assigned heliocentric motion to the planets which is the modern and correct view. Incidentally, a similar model was proposed by Tycho Brahe in Europe in late 16 century. Recently pioneering work by Ramasubramanian, Srinivas and Sriram [5] brought to light that a similar model of planetary system was given by Nilakantha of the Kerala school in 1500 AD.

(e) Calculational accuracy

Samanta had a passion for accuracy and precision in his computations. He has given the ratio of mean motion of planets per day up to ten places in sexagesimal system compared to his predecessors who restricted themselves to five places. He did not use the commonly used value $\frac{22}{7}$ for π : instead he took $\frac{3927}{1250}$ and also $\frac{600}{191}$. He invented a diagramatic, graphical method called *parilekha* for the calculation of eclipses. He has given 55 tables containing 50 numbers, each of which is needed to calculate final corrections.

(f) Instruments

Samanta developed a whole range of precision instruments for measurement and astronomical observations which were quite small, handy and simple. He has devoted a full chapter of *Siddhantadarpana* to the description of these instruments. Although these instruments are crude by today's standards, he could measure with them, and obtain values which are in close agreement with modern values, due to the skill acquired through long years of dedication and practice.

Samanta as Scientist

(a) Science for science sake

Samanta devoted his life to the study of astronomy for the sheer love of it, without seeking any return. The Viceroy of India, Lansdown invited him to be present in Calcutta on 28th August 1893, to confer on him the

title *''Mahamahopadhyaya''*. Samanta informed him regretting his inability to come, as it would interfere with his scientific pursuit, and daily worship. The Viceroy was so impressed with him that he organised the award ceremony a month later on 28th September at Cuttack, where the title was conferred by Commissioner Cook on his behalf.

(b) Courage of conviction

The courage of conviction shown by Samanta as a boy in his assertion that the old *siddhantas* like *Suryasiddhanta* and *Siddhantasiromani* are erroneous, is indeed a very rare quality to be found only in great scientists and thinkers.

(c) Dedication as a scientist

His dedication as a scientist is best described by Professor Jogesh Chandra Ray thus:

> This constant strain upon his body which had never been strong began to undermine his system. He contracted a disease which has been his constant companion. Dyspepsia with its attendant colic has impaired his health. At times it becomes so painful that he is compelled to break up conversation, roll down on the ground till the attack is over. Full meals, frugal as they are, he has not enjoyed for thirty years and has seldom permitted himself the indulgence of even half meal twice a day... Even in the present invalid state he would willingly sit up a whole night if it were for any thing connected with his favourite subject.

Recognition and Awards

(a) The Nature:

An extensive report on *Siddhantadarpana* was published by the International Journal, *The Nature*, in its 9th March issue of 1899 with the title ''A Modern Tycho''. In the final analysis it commented ''Prof. Ray compares the author very properly to Tycho. But we should imagine him to be greater''

(b) Knowledge:

The International Journal, Knowledge wrote in its report[6]:

> It is a complete system of astronomy founded on naked eye observation only. The work is of importance and interest to us westerners also. It demonstrates the degree of accuracy which was possible in astronomical observation before the invention of telescope.

(c) Award by Government of India:

The Viceroy of India, Lansdown honoured Samanta by conferring the title *Mahamahopadhyaya* in 1893.

(d) Award by Gajapati King of Puri:

In 1870, the Gajapati king of Puri conferred on Samanta, the title of *Harichandan Mahapatra* in recognition of his monumental work. A conference of Sanskrit scholars held at Puri in 1876 accepted the astronomical prescription of *Siddhantadarpana* for regulation of rites and worship at the Lord Jagannath temple, which has been continuing since then.

Conclusions

Samanta achieved the goal of detecting and correcting the errors that had accumulated in ancient Indian Astronomy over a long period. This was possible because of his unique discovery and measurements of all the three irregularities in the motion of the Moon, dramatic hiking of the distance of the Sun with respect to the Moon, his better model for the planetary system, fabrication of precision instruments and measurement of data of unprecedented accuracy and better methods of calculations up to ten decimal places. He has given guidelines for accurate prediction of eclipses and other celestial events for future generations. It is a living system of astronomy which is being used in Orissa for making almanacs and in Jagannath temple, Puri, for the regulation of rituals. It is in fact a complete system of astronomy founded upon naked eye observations as commented by the Journal, Knowledge. He is undoubtedly one of the greatest naked eye astronomers of the world, and it is fair to state that he rightly belongs to the same class of Indian astronomers as Aryabhata, Varahamihira, Brahmagupta, Bhaskara and Nilakantha.

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Persian Astronomical Tables Composed in India

Farid Ghasemlou and Negar Naderi

Introduction

Before introducing our main topic, a short remark about the word *Zij* seems necessary, because it will be used frequently in this article. *Zij* is derived from Persian word *Zig*. Changing G in Persian language (or as we say *Farsi*) to J in Arabic language has occurred so often. In Farsi *Zig* means thread, especially warp and woof of a carpet. Because of similarities between warp and woof of a carpet with astronomical tables, these books are named *Zij*. In Arabic its plural forms are *Aziya* and *Zijat*. This word has entered Greek and Latin languages as *Zich* or *Ezich*.

The Baburi dynasty or ''Great Mughals'' in India were interested in Persian language and literature; and a large number of scientists, poets and men of letters migrated to India. They wrote their works, naturally in Persian and adopted it as a vehicle of learning. One of the important styles of Farsi poems is named today the ''Indian style'', which is the heritage of that era. Iranian poets who had immigrated to India and were affected by the Indian culture, transmitted that Indian style into Iran. Among many scientific works in Farsi written by Indians, we have chosen the astronomical tables, in order to show the scientific exchanges between these two ancient countries.

A Survey of Islamic Astronomical Tables, composed by Prof. Edward Stewart Kennedy (USA), is the most important compilation of the *Zijes* of Islamic civilization. However, he has mentioned only two *Zijes* which were composed in India. In this paper we have tried to introduce a number of *Zijes* which are missing in Kennedy's *Survey.*

It is worthy to note that we know just one Persian *Zij* composed in India before the era of Baburids: the astronomical table of Wabaknawi[1].

[1] *Editorial note*: It seems that Wabaknawi's *Zij* was preceded by at least two Persian Zijes, known to-date; Al-Wabaknawi (Shamsuddin Muhammad bin Ali Khwaja Yamli) wrote in ca 720 AH/1320 AD a *Zij* in Persian with the title: *Al-Zij al-Muhaqqiq al-Sultani ʿala Usal al-Rasad al-Ilkhani* (The exactly

One of its manuscripts is extant in the Central Library of Tehran University. Among the *Zijes* composed after 16th century in Indian subcontinent, we have traced 11 of them in different manuscripts libraries in India, Iran, and other countries, The date of some of them could not be ascertained[2]. In addition to these, there are available some commentaries on a couple of these *Zijes*. Moreover there is also a *Zij* composed in Iran which shows influence of these *Zijes* composed in India. We introduce all of them here.

The dated Indian zijes

1. *Zij-i Muzafar Shahi.* This anonymous *Zij* was composed in the reign of Sultan Muzaffar Shah, the ruler of Gujarat, and dedicated to him. Though we do not know the precise date of writing of this work, yet it could be dated by the Muzaffar Shah's reign (917-932AH/1511-1525AD). That is, it was composed during his reign. So this is the first Persian *Zij* composed in India at the beginning of the reign of Babur. Its unique manuscript is extant in Lahore[3]. Dr. Aftab Asghar (of Lahore) has edited and published it[4].

2. *Zij-i Rahimi*: It was composed by Fariduddin Masckd Dehlavi in 1036 AH/-1627 AD. The author died before finishing this book and his

examined tables of Sultan, based on the principles of Ilkhani's observations). This title shows clearly the source of the Wabaknawi's *Zij*, i.e, Nasiruddin Tusi's *Zij-i Ilkhani* written in ca 1270 in Persian. In fact, Wabaknawi's *Zij* is a commentary on the latter. Three mss. of this *Zij* are known to exist presently: in Tehran, Ayasofiya (Istanbul) and in Mulla Firoz Collection in Bombay (India), see Sayili, Kennedy, and Storey. The preceding *Zijes* in Persian are: *Zij-i Asharfi* (ca. 1310) by Muhammad abi Abdullah Sanjar Yazdi and *Zij-i Nasiri* by Mahmud bin Umar and dedicated to Nasiruddin Mahmud bin Sultan Iltutmish, who ruled in Delhi during 1246-65. cf. Ansari (1995) pp.281-282.

Quite a substantial work on Indian *Zij*-literature has been done in the last decade. There is the paper by Maulana S.A.Khan Ghori in 1985. Following him, S.M.R. Ansari has published two studies: one in 1995 and another 1995/96 (see Bibliography). In the former he has dealt with briefly, besides others, most of the *Zijes* discussed in this paper. Since the authors are perhaps not aware of this material, we have inserted some informative footnotes at relevent places in this paper We are grateful to Prof. S.M.R. Ansari for this information and others in the sequel.

[2] Cf. the preceding works by Ghori (1985) and Ansari (1995).

[3] Cf. Munzavi (1983), p.278.

[4] See Asghar (1980), pp.113 *et seq.*

brother, Mulla Tayyab Ibrahim completed it. A unique manuscript of this work is extant in the Holy Shrine Library in Mashhad (Iran)[5]. This *Zij* comprises a preface and 4 chapters *(Maqala):* The author explains Hijra, Alexandrian and Persian calendars and the methods of converting the dates of one into others. In a separate section the author has also explained the Chinese *(Khatai)* calendar. This *Zij* consists of tables of sine *(jayb)* function, of the first and the second declination, and of the ascensions of the zodiacal signs for diverse latitudes. The last two tables are calculated for the latitude of Samarkand and some for the latitude of Akbarabad (Agra). In the meridian tables of different months besides these two cities, Lahore and Burhan are mentioned as well, and their meridian tables are also compiled. As the city of Samarkand has been chosen as the origin of the position tables for the observations of heavenly bodies, it shows that these tables are based on Ulugh Beg's astronomical table.

A large set of tables comprise the longitude and latitude of 448 geographical places. For them the origin or zero longitude is *Jaza'ir-i Khalidat* (Canary Islands), while the origin of latitude is the equator. The sources for these tables are *Takwim-al-Buldan,* composed by Abul Fida written in 721 AH/1321 AD and also Ulugh Beg's astronomical table. One finds in this *Zij* also a set of tables for the mean motion and apogee of the sun, for the equation of the sun, for equation tables for the first, second and the third equations of the moon in the twelve signs of the Zodiac, tables for the parallax of the moon, and tables for lunar and solar eclipses. An easy method for finding the solar ecliptic longtitude is also given.

3. *Zij-i Shah Jahani: Karanama-ye Sahib Kiran-i Thani;* composed by Fariduddin Masᶜkd b.Ibrahim Dehlavi, the author of the previous work. This *Zij* was composed during 1629-1630. Fortunately, a number of manuscripts of this *Zij* are extant: Asafiya Library in Hyderabad (India), in Leningrad(Russia), and in Bodleian Library in Oxford (Britian)[6]. It seems that the author has had a special liking for using similar phrases and sentences in his works. That is, the phrases in the beginning of these two astronomical tables (No.2 and 3), and also the first phrase of his another book: with the title *Siraj al-Istikhraj* (concerning astronomy), are the same. These common phrases/ sentences may probably mislead one who wishes to base his knowledge on just these first phrases.

[5] Catalogue of Astan-i Quds, vol 8, p.184.

[6] Storey, vol-2, part 1, p.89.

Zij-i Shahjahani (ZSJ) consists of a preface and 4 chapters *(Maqala)*. It differs with *Zij-i Rahimi* in that the preface of the latter has 4 parts, and the preface of the former has 5 parts. ZSJ are the oldest astronomical tables which give some information on *Ilahi* Calendar which was introduced by emperor Akbar and was employed also during Shah Jahan's reign. These tables were prepared under the patronage of Yamin al-Dawlah Asif Khan of Iranian origin who was a minister of Shah Jahan. The author has mentioned many times the names of both Shah Jahan and Asif Khan. This *Zij* comprises many tables for the first and the second declination including tables for correcting them, and tables for ascensions of the zodiacal signs etc. We may note that in the beginning of ZSJ and in the fourth part of its preface, all the 101 tables are listed[7]. These tables are compiled for Agra and Lahore (the latter was the birth place of Shah Jahan), and are also based on Ulugh Beg's astronomical tables.

4. *Zij-i Muhmmad Shahi Hindi* or astronomical tables of the Indian emperor Muhammed Shah. The author of this Zij is not known[8], and it was composed during the reign of Muhammad Shah (reigned 1719-1748). We know just one manuscript of it, which is preserved in National Museum in Karachi (Pakistan)[9].

5. *Zij-i Muhammad Shahi* (ZMS) or *Zij-i-Jadid Muhammad Shahi:* Its author is Raja Jai Singh Sawai[10] (d.1743) and it was composed during 1727/28 in the reign of Muhammad Shah. Fortunately there are many manuscripts copies of ZMS extant in the libraries of the world. In Iran there are manuscripts in the Library of the Holy Shrine in Mashhad, Library of the Parliament (Majlis), Malik Library and Central Library of Tehran University. This *Zij* had been very important, especially in Iran. In many annual calendars published more than hundred years ago, it has been mentioned that this *Zij* has been the basis of their own calendar calculations.

Because of its importance, many commentaries were written on it. We know presently five of them. The authors of these commentaries are

[7] Ghori has given the contents of ZSJ and also information regarding the author's objective in writing this *Zij*.

[8] Ansari has identified this *Zij* as a copy of *Zij-i Muhammad Shahi*, No.5 below.

[9] Munzavi, vol I, p.278.

[10] Ansari, who has prepared the *Critical edition* of ZMS, has concluded that ZMS was compiled by Mirza Khayrullah Muhandis, great grandson of Ahmad Ma'mar, the architect of the *Taj Mahal*.

Khayrullah Khan Muhandis 'Abdullah Khan bin Azimuddin Muhammad Khan (also known as Maharat Khan), Dust Muhammad Munajjim, Shar Muhammad Khan, and also an anonymous commentary. Some of these commentaries, as manuscripts are extant in the Holy Shrine of Mashhad, and of Majlis (Parliament) in Tehran. Shayr Khan's commentary is composed probably in 1736, i.e., (about 6 years after ZMS was composed). Manuscripts of commentaries by Khayrullah Khan[11] and Abdullah Khan bin Azimuddin Muhammad Khan are extant in Pakistan[12].

A certain Dust Muhammad Munajjim Khurasani in 1791/92 had tried to explain the complications and strange formulations of this *Zij* in his work titled by *Tashil-i Zij-i Muhammad Khani,* meaning, facilitating Muhammad Khani's astronomical tables. Two manuscripts of this commentary are known to exist in the library of the Theology Faculty of Tehran University. One of them is erroneously titled as *Zij-i Mohammad Shahi,* and not as its commentary. There exist two manuscripts of another commentary by an unknown author in the Library of Madrasa Sipahsalar in Tehran.

Zij-i Muhammad Shahi is the most important *Zij* that we could find. It informs us about Muhammad Shahi calendar, established during the reign of Muhammed Shah. For many years thereafter, this calendar was in actual use in Iran. Among the annual calendars in print in Iran in the past, the calendar of 1880 is the last one mentioning Muhammad Shahi calendar along with other calendars. After that year, this calendar was omitted from the Iranian calendars. Instead, the Irani calendar-makers employed the *Zij-i Bahadur Khani* as the basic *Zij* for the calculation of calendars in Iran. Like other *Zijes* ZMS contained also many tables, such as, tables for transforming different calendars into one another, tables for sine, tangent and cotangent functions, tables for the first and the second declinations, for ascension, for geographical longitudes and latitude of about 340 different cities relative to Canary Islands and the equator, elongation tables of planets, table for the visibility of Moon, and tables for peculiarities of stars. The last tables are based upon the parameters mentioned in *Zij-i IlKhani*; and annual variations in their peculiarities for the years 1725-1726. Thereby they supplement the information mentioned in *Zij-i IlKhani.*

[11] This particular commentary has not been found to-date. It has been only mentioned by Ghulam Hussain in his *Encyclopaedia of Sciences* (Jami^c Bahadur Khani). see Storey vol II, part 1, p.94

[12] Muzavi, Vol 1, p.277

Zij-i Muhammad Shahi had influenced deeply Mirza Baqir Mulla Bashi Mazandarani who composed a *Zij* in Persian. We do not know when and where this *Zij* was compiled, but it is clear that the author had used ZMS since he mentions it many times in his *Zij*. A unique manuscript of this Persian *Zij* is extant in Darulkutub Library, Cairo.

6. *Zij-i Mir Alami:* One manuscript of this *Zij* is in Asifiya Library in Hyderabad (India). It was compiled by Safdar Khan b. Muhammad Hussain Khan b. Muhammad Ismail Shirazi, and dedicated to the minister Mir Alam. We don't know the exact date of its composition, but as Mir Alam died in 1808, it must have been written before then.

7. *Zij-i Ashki:* One manuscript of this *Zij* is in Asifiya Library in Hyderabad (India). It was composed in 1816 by Raja Kundan lal Ashki b.Mannu Lal Dehlawi.

8. *Zij-i Bahadur Khani* (ZBK): It is also known by the names *Zij-i Bahaduri* and *Rasad-i Tughiyani*. This is the latest and the most important Persian *Zij* composed in India during 1841-1842 by Ghulam Hussain Jaunpuri. In Iran there are extant a few incomplete manuscript copies of this *Zij*. One of its manuscripts is kept in Asifiya Library in Hyderabad. Based on this manuscript, a lithograph was printed in 1855. We know that the author is a descendant of immigrants from Shiraz (in south of Iran) to India. As mentioned above, its another name is *Zij-i Tughiyani,* that is, named after a flood, because during the writing of this *Zij* a great flood had occurred. The author was deeply affected by it.

Ghulam Hussain had also other astronomical books in Persian to his credit, like the *Jami*ᶜ *Bahadur Khani* which is one of the most interesting Persian astronomical encyclopedias[13]. Although there is no manuscript of *Zij-i Bahadur Khani* in Iran, yet there are extant a few exemplars of the printed version of this book in public and private libraries. Along with other tables it contains diverse tables for peculiarities of stars, and tables for geographical coordinates of about 420 cities/localities with respect to Canary Islands and the equator. The author had been busy in writing astronomical works in Sahebganj in Bihar.

This *Zij* contains one perface and 7 chapters *(maqala)*. In the second chapter which is devoted to calendars, he has explained 16 different calendars, some of which are not found in other similar works/*Zijes*[14].

This *Zij* is counted as a very important compilation for its astronomical information, and it is in use in Iran even now. One of the

[13] For a recent paper on this *Encyclopedia*, see Ansari and Sarma (1999-2000).

Iranian experts in calendars, who died in 1999, has surveyed its influence, which will be published shortly.

The undated astronomical tables.

9. *Zij-i Talpar,* composed by Sultan Ali Baluch Khan known as Talpar. We know about just one unique copy, which is extant in Rashidiya school's Library in Islamabad (Pakistan). This copy was scribed in 1251 AH/ 1835 AD[15].

10. *Zij-i Nizami,* composed by Khwaja Bahadur Hussain Khan b. Bahadur Khan. One manuscript of this *Zij* is extant in Asifiya Library in Hyderabad (India). Probably, a commentary on it is also extant[16,17].

11. *Zij-i Hindi,* is a translation of an Indian work into Persian. The translator is Mirza Gul Beg Munajjim. According to him the title of the original work was *Hakindi.* We know just one manuscript, extant in National Museum in Karachi (Pakistan).[18,19].

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[15] Munzavi, vol. 1, p.276

[16] Storey, vol. 2, part 1, p.100.

[17] See Ansari (1995), p.283. This *Zij* was written in 1780, and 2 mss are extant in Hyderabad. It is actually based on the famous Indian astronomical tract, *Suryasiddhanta.*

[18] Munzavi, vol.1, p.279 . Another manuscript of this table is extant in the Raza Library in Rampur (India).

[19] For more details, see Ansari (1995), p.283 . It was written about 1805 AD.

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Islamic Astronomy in India during 16th-18th Centuries and its Interaction with Traditional Indian Astronomy

S. M. Razaullah Ansari

1. Introduction

In the context of the Kerala School of Astronomy during 14th –19th centuries and in which both Paramesvara and Nīlakaṇṭha emphasised particularly the observational astronomy, we sketch briefly the transmission of Islamic Astronomy [IA] into Medieval India, during 16th –19th centuries. To start with, we may mention that by IA we mean here the astronomy as developed during 9th–15th centuries in the Middle East (West Asia), Central Asia and Arabic-speaking North African countries; in short, astronomy as developed in the Islamic cultural areas during the medieval period.

From the outset a specific feature of IA was the activity to develop observational/ practical astronomy rigorously, which in turn led to the compilation of about 250 *Zījes*: the astronomical-mathematical tables used for the positional astronomy of the Sun, Moon, planets and then known 'fixed' stars. To that purpose, quite a number of famous observatories were established in Islamic countries. In the following we first enumerate this astronomical activity.

During the Abbasid Caliphate [al-Mansūr (754-775), Hārūn al-Rashīd (786-809), and particularly al-Mā'mūn (813-833) *et al.*] a rigorous observational activity was started at the early Islamic observatories at Shāmsīya and Qāsiyūn.[1] The programme was to collect accurate and systematic astronomical, geodetic and geographical data, for instance, to measure the length of the meridian-arc, determining the obliquity of the ecliptic, eccentricity of the solar orbit, position of the solar apogee etc.[2] As for the *corpus of Zījes*,[3] we may mention here the *Zīj-i al-Shāh*

[1] See the classical work on Islamic observatories by Sayili (1960).

[2] Cf. Ansari (1983), pp.43-44, table 1 for obliquity and table 2 for diverse observations.

[3] The classical paper on this corpus is by Kennedy (1956).

(translated from Pehlavi*), Zīj-i al-Khwārizmī and Zīj-i al-Mumtaḥan* (the Tested Tables) by Yaḥyā bin ᶜAlī;[4] they were fundamentally based on Indian Astronomy. However, the *Zījes* compiled later were based on the Ptolemaic Astronomy, for instance *Zīj-i al-Battānī* down to *Zīj-i al-Bīrūnī* and *Zīj-i al-Ḥakīmī* by Ibn Yūnus (d.1000).[5] Somehow or other these *Zījes*, written in Arabic, were not transmitted to India during the Medieval period. On the contrary, *Zījes* written in Persian and compiled during the 13th–15th centuries were brought by the scholars who thronged the courts of Sultans and Mughal emperors, and who came from Iran and Central Asia. They are: *Zīj-i Ilkhānī* by al-Ṭūsī (d.1274),[6] *Zīj-i al-Khāqānī* by al-Kāshī (d.1413), and the well known *Zīj-i Ulugh Beg* (completed in 1437/38).[7] We have dealt with the transmission of this *Zīj*-literature into India and its influence on the development of astronomy in Medieval India in sufficient detail elsewhere.[8]

2. Islamic practical astronomy in India

2.1 Following the example of the rulers in West-Central Asia, the following Indian Sultan and Emperors commissioned and sponsored the establishment of observatories:[9]

- Sultan Fīrūz Shāh Bahmanī (reigned 1397–1422), at Balaghāt in 1407,
- Humāyūn (reigned 1530–36) at Delhi,

[4] Both al–Khwārizmī and Yaḥyā bin ᶜAlī were al–Mā'mūn's astronomers of 9th centuries.

[5] al-Battānī (d. 929), al–Bīrūnī (973–1048) whose treatise on astronomy consists of both practical and theoretical astronomy, including the Indian Yuga–astronomy. Michio Yano (Kyoto) is preparing an edition of al–Bīrūnī's Qānūn al–Masᶜūdī.

[6] al–Ṭūsī was the director of the astronomical observatory located at Marāgha, in present day Iran. He is particularly famous for his critique of Ptolemaic astronomy. For the latter, see the Essay Review by Ansari (1994).

[7] Sultan Ulugh Beg bin Shāhrukh (1393–1449), ruler of Samarqand (now in Uzbekistan) got an astronomical observatory built in 1420 for the specific purpose of compiling a *new Zīj–i Jadīd*. For details see Ansari (1995), p.293, and Sec. 3.1 below. For his observatory see Sayili (1960), the relevant chapter.

[8] Ansari (1995), pp.281 *et seq.*,wherein are especially mentioned the presently available manuscripts holding of the Indian libraries, Appendix I, esp. pp.290, 292–293.

[9] See details in Ansari (1997/2001), Sec. 3.1, pp. 60–62.

- Shāh Jahān (reigned 1627–58) at Delhi, under the directorship of Mullā Maḥmūd Jaunpūrī (1606–1652),
- Muḥammad Shāh (nominal ruler during 1719–48), under the supervision of Raja Sawai Jai Singh II (1686–1743). The observatories were built in Delhi, Mathura, Benares (Vārānasī), Ujjain and Jaipur, and are called in folk parlance: *Jantar Mantar.*[10]

2.2 In the field of Zīj-literature, the scholars of Medieval India were also not lagging behind their colleagues of West-Central Asia. The following list indicates amply the vigorous activity in this branch of practical astronomy.[11]

1. *Zīj–i Nāṣirī* , dedicated to Naṣīruddīn Maḥmūd bin Sulṭān Shamsuddīn Iltutmish (ruled in Delhi 1246-1265), the author being Maḥmūd bin ᶜUmar (Ms. in Tabriz / Iran).

2. *Zīj–i Jāmiᶜ Maḥmūd Shāhī Khiljī*, compiled during 1438-1460 by an anonymous Indian scholar (Ms. in Oxford), dedicated to Sultān Maḥmūd Shāh Khiljī (reign 1435–1469).

3. *Tashīl Zīj–i Ulugh Bég,* a commentary on Ulugh Beg's Tables (ZUB) by Shaykh Chānd ibn Bahāuddīn, the court astronomer of emperors Humāyūn and his son Akbar (reign 1556–1605). To note is that Akbar ordered during his reign the translation of ZUB into Sanskrit, and which was carried out by a team of scholars. One of its copies is extant in the City Palace Museum of Jaipur.[12]

4. *Zīj–i Shāhjahānī*, dedicated to emperor Shāhjahān, compiled by the court astronomer Farīuddīn (d.1630).[13] The Emperor's Hindu court astronomer Nitynanda translated it into Sanskrit. One copy of this translation is in the City Palace Museum (Jaipur).[14]

5. *Zīj–i Muḥammad Shāhī* (ZMS), got compiled by Maharaja Sawai Jai Singh and dedicated to emperor Muḥammad Shāh has been widely known.[15] This is the most important *Zīj* of the Medieval India. In fact it

[10] *Ibid.*, p.62. For more details see the important monograph by Sharma (1995).

[11] Ansari (1997/2001), pp. 62–64. For the first report, see Ghori (1985), cf. also Ansari (1995), pp.227, 281–282. See also the contribution of Ghasemlou and Naderi in these Proceedings.

[12] Cf. more about it in Sec.3.1.

[13] Ghori (1985), pp.34–36, see also the paper of Ghasemlou and Naderi in these Proceedings.

[14] See further regarding this translation etc., in Sec.4.2.

[15] Cf. Ghori (1985), pp.36–41, Ansari (1995), p.283.

replaced throughout Islamic countries even the standard *Zīj-i Ulugh Bég* prepared in the 15th c. at Samarqand. ZMS was compiled by Mirzā Khayrullh Muhandis (d.1747) who belonged to a distinguished family of mathematicians. It may be added that the significance of ZMS lies in the fact that the author synthesised therein the Central Asian *Zīj*-tradition with the European astronomical tables of Phillipe de la Hire (d.1718).[16] These Tables were based on the most accurate astronomical observations carried out with the help of the telescope furnished with micrometer and cross-wires. In passing, it may be added that this author has prepared a critical edition of the Persian text of ZMS, to be published shortly with a long introduction in English on the specific *problématique.*

2.3 For the sake of completeness, we may enumerate here seven *Zījes*, complied during 19th century. Some of them are based on (or adapted from) Indian *Siddhāntas* and *Karanas*, while others are in the style of the Central Asian *Zījes*.[17]

1. *Zīj-i 'Safdarī* , by 'Safdar ᶜAlī Khān, written in 1819, Mss. in Hyderabad (India). It is a Persian translation of a Sanskrit work *Grahacandrika* (planetary and lunar tables). It may be identified with the *Grahacandrika Ganita,* by Appayya, son of Marla Perubhaṭṭa. 'Safdar ᶜAlī Khān also wrote in ca 1808 a first draft of this *Zīj*, with the title: *Zīj-i Mīr ᶜĀlamī*, its manuscript is also extant in Hyderabad (India).

2. *Zīj-i Sarkūmanī*, Persian translation of *Siddhāntasiromaṇī* of Bhāskara II (written in 1150 A.D) by 'Safdar ᶜAlī Khān, who mentions in No.1 above that he translated it in 1797. No copy of this translation has been found to-date.

3. *Zīj-i Hindi* by Gulbeg Munajjim bin Muhammad ᶜAlī Riyāḍī , written about 1805, based on Sanskrit tables by Makaranda (ca.1478), the source of which was *Sūryasiddhānta.* Its manuscripts are extant in the manuscripts collections of Raza Library (Rampur), and National Museum of Pakistan (Karachi).

4. *Zīj-i Sulaymān Jāhī* by Sayyid Rustam ᶜAlī Raḍwī, and dedicated to the King of Avadh—Naṣīruddīn 'Haydar (reign 1827–1837). This *Zīj*

[16] *The Tabulae Astronomicae* were published first in 1687, the second edition in 1702, which was reprinted in 1727. It was brought to Maharaja Jai Singh by the Jesuits delegation, which was sent to Portugal by the Maharaja in 1727. For a recent study, see the paper by Benno van Dalen (2000).

[17] Ansari (1995), pp. 283–284; Ansari (1997/2000), pp. 70–71. Cf. also Ghasemlou and Naderi in these Proceedings.

seems to be an adaptation in Persian from *Bhāsvatī* in Sanskrit, the author of which may be identified with Śatānanda (fl. ca. 1099).[18]

5. *Zīj–i Ashkī*, by Kundan Lāl Ashkī son of Mannū Lāl Falsafī, written in 1816, autographed manuscript of 62 pages in Hyderabad. This *Zīj* is written in the traditional style.

6. *Zīj-i Bahādurkhānī*, by Ghulām Ḥusayn Jaunpūrī, written in 1838 and printed in 1855 in Benares.[19] It is largely based on *Zīj–i Muḥammad Shāhī*. Evidently it follows the style of Central Asian *Zījes*.

3. Islamic astronomy in early Sanskrit sources

There has been ample interaction between Arabic-Persian knowing astronomers-mathematicians and the Sanskrit *Jyotisha* experts even upto the 19th century. We may note the following particular instances.

3.1 During Emperor Akbar's reign, a team of scholars were commissioned to translate *Zīj-i Ulugh Beg* (ZUB).[20] This translation was often cited in the 18th century at Benares by the astronomers who supported the Islamic mathematical astronomy as against their critics: the *siddhāntists*.[21] An 18th century manuscript of this translation is extant at the City Palace Museum Library, Jaipur. It was acquired from the city of Surat through Nandarm Joshi.[22] The tables of ZUB are transliterated into *Nāgarī* in this copy.

3.2. As a matter of fact, Islamic Astronomy had already been introduced into Sanskrit sources long before the Mughals. We may recall here the *first* book in Sanskrit on astrolabe: *Yantrarāja* authored by Mahendra Sūrī, the court astronomer of Sulṭān Fīrūz Shāh Tughlaq (reign 1351–1388).[23] A commentary on this text by his pupil Malayendu is also well known. Besides treating the art of construction of astrolabe and its use, Sūrī had employed in this work the Islamic astronomical-mathematical parameters freely. They are: [24]

[18] Sen *et al*, p.193.

[19] See Ghori (1985), pp.42–44; Ansari (1995/96); cf. also Ghasemlou and Naderi in these Proceedings.

[20] Reported *first* by the chronicler at Emperor Akbar's court: Abul Fazal (d.1602), see Blochmann (1927), 'īn 34, p.110.

[21] Pingree (1978), p.320.

[22] Bahura's *Catalogue* (1971), pp.58-59, Ms. No.45.

[23] Ohashi (1997), pp.210–216.See also the recent paper by Sarma, S.R.(2000).

[24] Pingree (1978), p.318.

- Sine tables with R=3600= 1,0,0;
- Tables of declination with the Islamic value of obliquity 23.35° as against the usual Indian value of 24°. Al-Battānī (epoch 880 AD) determined this value, which was used or re–determined by several of his successors: Ibn Amājur, al-Khāzin (ca. 970), Abul Wafā' (ca. 977), Ibn Yūnus (1003), Al-Bīrūnī (1007 and 1019) etc.[25]
- Use of Ptolemaic system of geographical co-ordinates for the cities;
- Co-ordinates of 32 astrolabe stars derived from Ptolemy's *Almagest* for which the precessional longitude correction was taken at the rate of 1° in 66 ⅔ years for the year 1370.

Sūrī was a pioneer in introducing astrolabe to the Sanskritists.[26] A large number of similar monographs on astrolabe were written in the following centuries.[27]

4. Influence of Islamic astronomy during 16th–18th centuries

4.1 One of the Kerala astronomers, Acyuta Piṣāraṭī (1550–1621),[28] whose teacher Jyeṣṭhadeva (1500–1600) is the author of the famous work *Yuktibhāṣā*, is known particularly for his formula to reduce the mean longitude of the moon to the ecliptic longitude. This formula is enunciated in his work: *Sphuṭanirṇaya* (written ca.1593).[29] Now according to Pingree,[30]

> Such a reduction had first been suggested by Yaḥyā ibn Manṣūr in his *Zīj*

[25] Ansari (1983), Table 1, p.43.

[26] The Sanskrit text of his monograph has been printed, see for its summary and references in Ohashi (1997).

[27] Ohashi (1986–87). Later Ohashi (1997) introduces details of two more texts by Padmanābha (1423), and Rāmacandra (1428). For a general introduction to this topic see the paper of Sarma, S.R. (1999), where one finds the most recent findings on this subject matter.

[28] For his biographical sketch, see Sarma, K.V.(1972), pp.64–65, and Raja (1963), pp. 158–159.

[29] Sarma K.V.(1972) gives this formula, see p.13. He further states that the rationale of this correction is elaborated in Acyuta's work: *Rāśgolasphuṭanīti*, ibid p.64, the text of which has been studied by Sarma, K.V. (1955) thoroughly.

[30] Pingree (1978), p.319.

Editorial note: It may however be noted, as has been done in section 1 of this paper, that *Zīj-i al–Mumtaḥan* itself was composed under the influence of Indian astronomy.

al–Mumtaḥan composed under [Caliph] al–Mā'mūn in 820s. Acyuta probably bears witness to some transmission of at least a part of Islamic Lunar theory that took place on the Malabar coast in the fifteenth or sixteenth century.

We may comment here by adding that transmission of this sort can be ascertained by a comparative study of Acyuta's formula and the resulting lunar tables with those of Yaḥyā in his *Zīj*, which is now available in print[31]. A study of his lunar theory has been carried out by Salam and Kennedy.[32]

4.2 As mentioned in Sec. 2.2, during the reign of Shāh Jahān (1627-1658), his court astronomer, Farīduddīn Masᶜūd bin Ibrāḥīm Dehlawī (d.1630), compiled the *Zīj-i Shāhjahānī* (ZSJ), which was completed in 1629/30. This is the first Indian *Zīj* which was translated in 1630 into Sanskrit by the Hindu court astronomer of Shāh Jahān, Nityānanda, for the Emperor's minister, Āṣaf Khān. The title is *Siddhāntasindhu*.[33] No study of this 'enormous',[34] very important Islamic astronomical text in Sanskrit has been done so far. Nityānanda wrote also another astronomical treatise in Sanskrit: *Sarvasiddhāntarāja* (SSR), in 1639.[35] We summarise below from the recent studies of the SSR by Pingree.[36]

1. Islamic mathematical astronomy is transformed into the framework of Indian Yuga Astronomy, i.e., conversion of mean motion of planets, and longitude of their apogees and nodes into integral numbers of rotations in a *Kalpa* , with corrections *(bījas)* wherever necessary. Thus, the mean longitude of planets calculated according to the

[31] The text of the *Zīj* has been published in *facsimile* by Fuat Sezgin (Frankfurt, 1986).

[32] Salam and Kennedy (1967).

[33] Four manuscripts copies of this translation are extant in Jaipur Manuscripts Collections: three in *Khāsmohor* , one in the Palace Museum (Ms.No.23, see below). However, the latter carries the title: *Zīca Nityānandī Shāhjahānī*. For the mss. in *Khāsmohor* collection, see Bahura (1976), p.124.

[34] The Jaipur Museum Ms. No.23, is of size: 44.6×31.5 cm, with 444 folios, scribed by Gaṅgārāma in 1727 for Raja Sawai Jai Singh, see Bahura (1971), p.58, and also its short description on p.96.

[35] On Nityānanda, see Sen *et al*, p.159, wherein the manuscripts of *Siddhāntasindhu* are not listed.

[36] Pingree (1978), pp.323–326; Pingree (1981), p.30; Pingree(1996), pp.476–480.

methods of ZSJ agree with those calculated by using Yuga astronomy.[37]

2. Method of computation of longitude and latitude of celestial objects using tables of ZSJ, the underlying theory of which is in the Ptolemaic–Islamic tradition. According to Pingree, "the [planetary] models themselves are thoroughly Islamic, with equants, protective spheres, and crank–mechanisms for the Moon and Mercury. The cosmology is almost equally Islamic".[38]
3. Tabulation of Sine of angles between 1° to 90°(for R = 60) to 5 sexagesimal places in steps of 1°, also with a table in steps of 1 minute of arc ; rules for their computations according to al–Kāshī [39] and ZUB, i.e., respectively for sin 1° and sin 1', are also included.[40]

We may add that the determination of sin 1° = 0.017452406437283571 by Jamshīd Ghiyāthuddīn al–Kāshī by solving a third degree equation using his original iterative method is described in his tract: "On the Chord and Sine", which was quite well known in Samarqand in 15th century and thereafter.[41]

4.3 The fact, that the knowledge of Islamic Astronomy, its terminology, methods and even its underlying cosmology was quite prevalent during the period in question, is also corroborated by other writings. Nṛsiṁha (fl. ca. 1586) and Munīśvara (ca..1646) criticised the Islamic concept of crystalline *(kāca)* spheres, and Kamalākara s/o Nṛsiṁha (ca.1616) employed the prime meridian passing through "Khāladātta" and thereby counting the "tūl" of Laṅkā as 112°. In Islamic geography *al-Jazā'ir al–Khāladāt* (the Canary Islands) were used for the zero-meridian following Ptolemy's Geography, and the term used for geographical longitude was *'Ṭūl al–Bulad.* Kamalākara indicates his acquaitance with the table of sine function after "Mirjolukabega", i.e., with *Zīj-i Ulugh Beg* and he seems to be the first Sanskrit scholar who

[37] Pingree(1996), p.479.

[38] *Sarvasiddhāntarāja*, *3*, pp.197 sqq.; *3*, pp.180–196, as cited by Pingree (1978), p.325.

[39] Al–Kāshī (d. 1429) was the director of Ulugh Beg's Observatory at Samarqand.

[40] Pingree (1978), pp.325–326.

[41] The Arabic title is: *Risāla al–Watar wa'l-Jayb*. A number of commentaries were written on this tract; see details in the entry on al-Kāshī, in *DSB*, Vol. VII, pp.255–262.

includes a chapter (*Adhikāra)* on geometrical optics in his *Siddhānta-tattvaviveka*, written about 1658.[42]

The well-known short tract on IA by ᶜAlī Qushchī (d. 1474, and compiler of ZUB) was translated as *Hayathagrantha* (to-date available mss. copied in 1694, 1730, and 1764[43]). The Persian title is *Risālah dar Ha'yat*; the Arabic/Persian word *Ha'ya(t)* connotes principles of astronomy. This tract was very popular in all Muslim *madrasas* till the first quarter of 20th century. The Persian text was lithographed a number of times in India (Delhi 1874,1898 etc.).[44] In the following century, a whole series of translations of important texts from Persian (and also from Arabic) into Sanskrit was initiated and sponsored by Maharaja Sawai Jai Singh (1686–1743).[45] We may mention here the most important translations: [46]

Al–Birjandī's Commentary on *Tadhkirah fī ᶜIlm al-Ha'ya* (Memoir on Astronomy) by Naṣīruddīn Ṭūsī (d.1274)[47] and also Ṭūsī's recension of Ptolemy's *Almagest.*[48] The latter was rendered into Sanskrit by Jagannātha (b.1652) as *Samrātsiddhānta.*[49] The former was translated into Sanskrit by Nayanasukhopādhyāya (fl. ca. 1730).[50] This Sanskrit translation consists of that chapter of Ṭūsī's *Memoir*, wherein he describes his own planetary model[51] in contradistinction to that of Ptolemaic, and that part of Birjandī's commentary, wherein the *Critique* of the Marāgha

[42] Pingree (1978), pp. 321–322, where the original references are quoted.

[43] *Editorial Note*: This has been published by the Sampurnanand Sanskrit University. *Hayatagrantha*, V.Bhattacharya, SBG 96, Varanasi, 1967.

[44] For details of ᶜAla'uddīn ᶜAlī Qushchī's works extant in India, see Ansari (1995), p.293.See also the relevant entry in *DSB.*

[45] It is known that the Muslim assistant of the translators was Muḥammad Ābid, Sarma, S. R.(1998), p.78.

[46] Pingree (1978),p.328, Pingree (1996), p.483–484.

[47] Ansari (1995), pp.289–290. Ṭūsī was the founder and director of the Observatory at Marāgha (now in Iran). For this very important observatory, see Sayili (1960), relevant chapter.

[48] Ansari (1995), p.291, Sec. 11(5).

[49] See Sen *et al* (1966), pp.89–90, for details of this *Siddhānta* and on Jagannātha. He is said to be the teacher of Raja Jai Singh.

[50] On him, see Sen *et al* (1966), p.153, wherein this translation is not mentioned.

[51] This is by now the famous "Tusi–couple", a term coined by Kennedy, cf. Kennedy (1966), esp. pp. 368–371.

school of astronomers headed by Ṭūsī is treated.[52] The fact, that the astronomy of the Central Asian school of astronomers, who were critical of the Ptolemaic astronomy, had been made available by Jai Singh to his school of astronomers, should be appreciated from the point of view of the creative assimilation of IA by Jai Singh.

5. Concluding remarks

Creative assimilation through the medium of translation in the first instance, is in actuality a prerequisite of a nascent scientific revolution in the world of ideas. Jai Singh had probably that in mind and he might have also hoped that the dynamics of new ideas, borrowed from Islamic cultural areas, would open up new dimensions for the reform of Indian theoretical astronomy and mathematics.[53] However, the introduction of European science/astronomy into the Indian subcontinent by Jesuit geographers-astronomers created another kink in the development of astronomy in Medieval India. It goes to the genius of Jai Singh that he welcomed it with scientific spirit, and even permitted an unbiased compilation of his tables, the *Zīj–i Muḥammad Shāhī*, on the basis of the tables of the Frenchman, Phillipe de La Hire.[54]

To conclude, we may emphasize, that emulating Maharaja Jai Singh our present day scholars should study in detail the various primary sources, particularly of the late Mughal period, in order to work out the extent and implication of transmission of IA and also of European astronomy on the Indian subcontinent.

[52] See Ansari (1994), II, pp.271–273, for a summary of this *Critique*.

[53] He was the *first* to order translation of Euclid's *Elements* into Sanskrit, the *Rekhāgaṇita*, from the Arabic recension of al–Ṭūsī (*Taḥrīr Uṣūl li–Uqlīdis*), written in the 13th century. The Sanskrit text is printed, see Sarma, S.R. (1998), p.75. for details.

[54] For recent works, see Pingree (1999) and Benno van Dalen (2000), cf. also Ansari (2001).

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Nilakantha's Geometrical Demonstration of Sums of Series of Natural Numbers

V. Madhukar Mallayya

Introduction

The classical Indian language Sanskrit is rich in technical literature and in addition to the vast mass of literature on philosophy, religion, poetry, drama etc., there exists a large store of technical literature bearing on subjects like architecture, iconogrophy, music, dance, astronomy, mathematics, medicine, and so on. The fund of knowledge embedded in the vast store of writings on such technical subjects did not spring up all of a sudden in a day or two, but was acquired through ages and traditionally handed down from the learned *acaryas* to their disciples in succession and thus clings by its roots running deep into the earliest strata of Indian civilization. The rich and complex Indian civilization flourished with a good scientific and technological base and the enormous mass of technical literature in Sanskrit containing the art and science treasures is a rich legacy that ancient and medieval India has left to the modern world. Study of Sanskrit texts on the twin sciences of astronomy and mathematics serves as the best key to open the science treasures of India lying hidden in them. A detailed and critical study of the contents of these texts will unfold the tremendous achievements of the ancient scholars which will serve as a source of inspiration to the present and future generations to make their own contributions in this field of knowledge.

In the field of astronomy and mathematics, India has produced a galaxy of erudite scholars who have contributed their invaluable share for enriching the twin sciences. Among them Gargya Kerala Nilakantha Somayaji (or Somasutvan) of Kerallur Illam in Trikkandiyur Village (near Tirur in Ponnani of Malappuram District in Kerala) occupies a prominent place. This versatile scholar who lived during the period 1444-1545 AD has authored several works dealing with astronomy and mathematics which includes *Tantrasangraha, Aryabhatiyabhashya, Golasara, Siddhantadarpana* and an auto-commentary on it, *Grahanadigrantha, Candracchayaganita* and a commentary on it, *Grahananirnaya, Sundararaja-prasnottara, Grahaparikshakrama* and the *Jyotirmimamsa.*

The *Aryabhatiyabhashya* of Nilakantha is an elaborate commentary on *Aryabhatiya* of Aryabhata. Some interesting geometrical derivations of various mathematical results are provided in this commentary written about a millenium after the composition of the basic treatise *Aryabhatiya.*

While the diagrammatic representation of algebraic and arithmetic truths is as old as geometry itself, the geometric representation of series is a special feature of Indian mathematics. The developement of geometric representation of series using piles (*citi)* is probably inspired from the construction of altars for the performance of vedic rituals. The use of terms *citi* or *upaciti, citighana, vargacitighana,* and *ghanacitighana* in *Aryabhatiya* for sums of certain arithmetical series is an indication of a sound knowledge of the intimate relationship of series with geometry during those days. The representation of series using rectangular strips introduced by Nilakantha and followed by his disciple Sankara is perhaps motivated by Aryabhata's terminology like *citi, upaciti* etc.

Sum of arithmetic series

Nilakantha represents the terms of an arithmetic series by rectangular strips whose lengths are set equal to the magnitudes of the successive terms and breadths equal to unity (see Fig. 1).

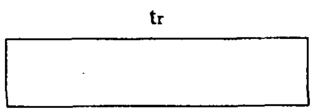

Fig. 1: r^{th} term of the series

The rectangular strips representing the successive terms of the series are piled up side by side (see Fig. 2) in the order of succession, and the

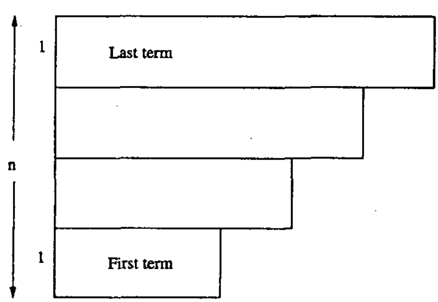

Fig. 2: *Sredhikshetra* representing the n successive terms of an arithmetic series whose sum is S_n.

figure so formed is called a *sredhikshetra* or a series figure whose area gives the sum S_n of the series, where n is the number of terms in the series.

Two identical *sredhikshetras* representing S_n are joined together (by arranging the second one in the reverse order), to form a bigger rectangular strip whose length is equal to the sum of the first and the last terms and breadth equal to the number of terms in the series. (see Fig. 3).

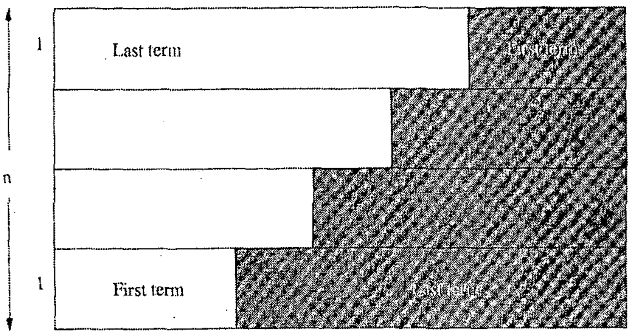

Fig 3: Two *Sredhikshetras* joined together whose area is equal to twice the sum of the series, $2S_n$

The product of the number of terms in the series n, with the sum of the first and the last terms will be equal to $2S_n$ so that half of the product gives S_n. Thus $S_n = \frac{n}{2}(a+l)$, where a is the first term, l is the last term or n^{th} term of the series. The above representation is exactly equivalent to the geometrical demonstration of the mathematical procedure given below.

Consider an arithmetic sequence:

$$(a,\ a+d,\ a+2d........,\ a+(n-1)d)$$

The sum of this given by

$$S_n = a+(a+d)+(a+2d)+............+[a+(n-1)d].$$

Arranging the terms of the series in the reverse order,

$$S_n = [a+(n-1)d]+[a+(n-2)d]+[a+(n-3)d]+.............+a,$$

and adding them

$$2S_n = \{\ a+[a+(n-1)d]\ \}n = (a+l)n,$$

so that

$$S_n = \frac{n}{2}(a+l).$$

If $S_n = 1+2+3+...............+n$, then $2S_n$ is the rectangular strip whose length is $1+n$ and breadth n formed by joining two identical *sredhikshetras* representing $1+2+.....+n$ after inverting one of them. Thus $2S_n = n(n+1)$, or

$$S_n = \frac{n(n+1)}{2} = \frac{n}{2}(n+1)$$

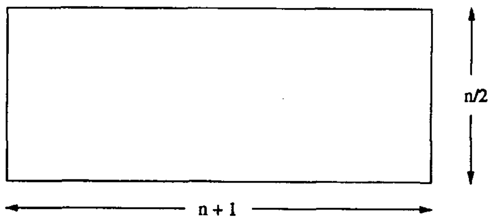

Fig. 4: Half of the rectangular strip obtained by joining two *Sredhikshetras* together, whose area is equal to the sum of first n natural numbers

Sum of sums

In three dimensions, a rectangular slab of length $\frac{n}{2}$, breadth $n+1$ and thickness 1 has the volume (see Fig.5),

$$S_n = \frac{n}{2}(n+1) = 1+2+3+............+n.$$

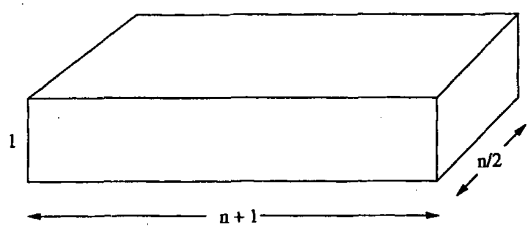

Fig 5: A rectangular slab representing the sum of first n natural numbers, S_n. This is the same as Fig. 4 but with thickness 1 unit

To validate geometrically the formula for sum of sums of natural numbers, viz.,

$$S_1 + S_2 + \ldots\ldots\ldots\ldots + S_n = \frac{n(n+1)\ (n+2)}{6},$$

where $S_r = 1+2+3+\ldots\ldots\ldots\ldots+r$, Nilakantha first constructs geometrical figures as shown in Fig.5, for all the different sums $S_1, S_2, \ldots\ldots\ldots\ldots, S_n$ which are found to increase gradually in size. Each figure is a rectangular slab of sides half-period and period + 1 with thickness 1. Suppose the period or number of terms is n . Then with six sets of slabs representing S_i, $(i=1,\ldots\ldots\ldots\ldots,n)$ a rectangular block of sides n, $n+1$, and $n+2$, can be constructed as described below (see Fig.6).

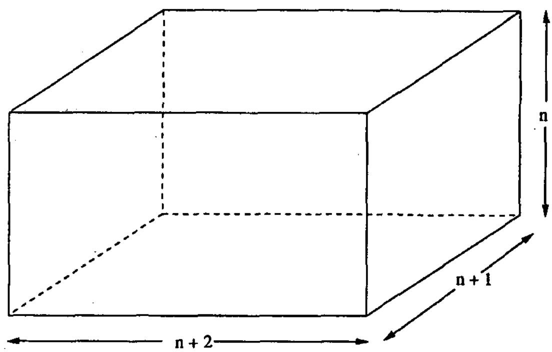

Fig. 6: A rectangular slab of length n+2, breadth n+1 and height n.

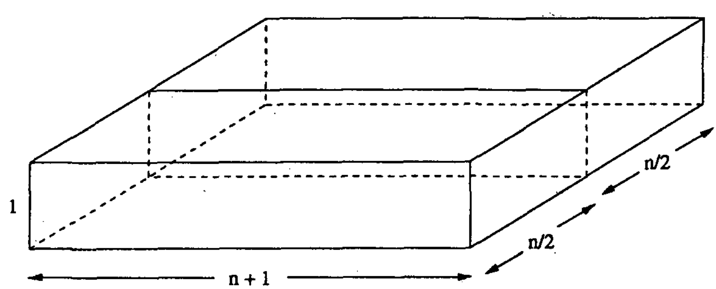

Fig. 7: Two S_n blocks forming a rectangular slab of sides n, n + 1, and thickness 1

(i) Two slabs representing S_n are joined together to form a bigger rectangular slab of sides n, $n+1$, and thickness 1 (see Fig.7).

(ii) With six such slabs representing S_n, three bigger rectangular slabs of sides n, $n+1$, and thickness 1 are formed.

(iii) Out of these three slabs, one slab is placed on the floor horizontally the second one is erected on its thickness with its length coinciding with the length of the first and breadth vertical as a wall and the third slab is placed on its thickness in such a way t at its breadth is vertical and its length coincides with the line formed by the breadth of the first and thickness of the second.

(iv) This gives a floor of sides n, $n+1$, and thickness 1 with walls of height n externally, and $n-1$ internally, having thickness 1 on the two adjacent sides.

The figure so formed called *kuttima* will have a hollow interior of dimensions $n+1$, n, $n-1$, with its walls and floor having thickness 1. The external length, breadth and height of the figure are $n+2$, $n+1$, and n respectively (see Fig. 8).

Now using six figures representing S_{n-1}, three rectangular slabs having length n, breadth $n-1$, and thickness 1 are formed. Two of them are placed on their thickness in the hollow interior of the former *kuttima* in such a way that their breadths are vertical touching the walls of the

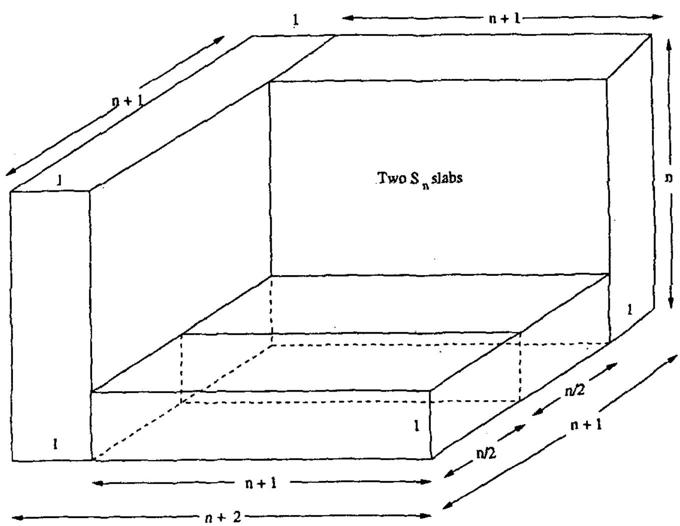

Fig. 8: *Kuttima* formed from six S_n slabs

kuttima and the third slab is placed horizontally on the floor of the *kuttima*. Now the interior hollow part has length, breadth and height $n,\ n-1,\ n-2$, respectively. It can be seen that the thickness of the floor and walls have increased by one unit and the length breadth and height of the interior have decreased by one unit. Similarly, using the rectangular slabs representing S_{n-2}, the thickness of the floor and walls will be increased again by one unit and consequently the length, breadth and height of the interior are all decreased by one unit. Proceeding like this with six rectangular slabs for each of the sums S_{n-3}, $S_{n-4},\ldots\ldots\ldots,S_2,\ S_1$, the hollow interior will be completely filled and a full rectangular solid is obtained, whose floor thickness is n which is now just the height of the solid. The length and breadth of this solid formed are $n+2$ and $n+1$. The volume of this rectangular solid is

$$6S_n + 6S_{n-1} + 6S_{n-2} + \ldots\ldots\ldots\ldots + 6S_1 = n(n+1)(n+2).$$

Hence,

$$S_1 + S_2 + S_3 + \ldots\ldots\ldots\ldots + S_n = \frac{n(n+1)(n+2)}{6}.$$

Sum of squares of natural numbers.

Now we give the geometrical demonstration for the formula for sum of squares of the first n natural numbers, viz.,

$$1^2 + 2^2 + 3^2 + \ldots\ldots\ldots\ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

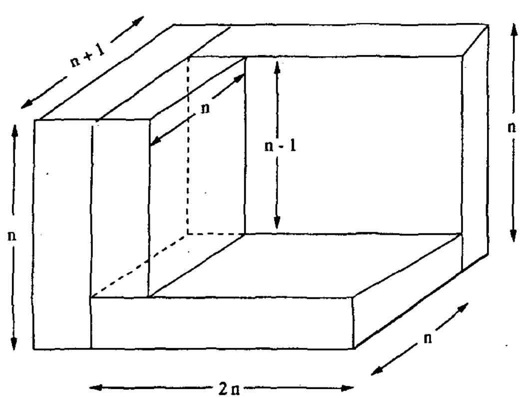

Fig. 9: *Kuttima* formed from six square slabs of sides n.

Consider six square slabs of side 1, six square slabs of side 2, etc., and six square slabs of side n, all of thickness 1. From the six square slabs of side n and thickness 1, we can construct two slabs of sides $2n$ and n, one slab of sides $n+1$ and n, and one slab of sides n and $n-1$, all of thickness 1. These can be arranged to form a *kuttima* as shown in Fig.9.

The external length, breadth and height of the *kuttima* are $2n+1$, $n+1$, and n. The hollow interior has dimension $2n-1$, n and $n-1$. We can again rearrange and place the six square slabs of side $n-1$, and thickness 1 inside this hollow, and so on, till we have exhausted all the sets of six slabs. Finally, we have a solid rectangular block of sides n, $n+1$ and $2n+1$. Thus using six figures each representing $1^2.1$, six figures each representing $2^2.1$, etc., and six figures each representing $n^2.1$, a rectangular block of sides n, $n+1$, and $2n+1$ may be generated. Hence,

$$6.1^2.1+6.2^2.1+........+6.n^2.1 = n(n+1)(2n+1)$$

which gives the sum formula:

$$1^2+2^2+3^2+................+n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Sum of cubes

To demonstrate the formula for sum of cubes, *viz.*,

$$1^3+2^3+3^3+.............+n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

in a geometrical way, Nilakantha first represents $\left(\frac{n(n+1)}{2}\right)^2$ geometrically as a square slab of sides $\left(\frac{n(n+1)}{2}\right)$ and thickness 1 and then cuts off a gnomon of width n from this (see Fig.10).

This gnomon is then cut into smaller slabs of successively decreasing sizes, starting from the corner in such a way that the corner is a square slab of side n and thickness 1, and the remaining slabs on both sides have one of their sides lessened successively by one unit so that they are $n-1$, $n-2$,.........,2, 1. Thus from the gnomon, a corner square slab having dimensions (n, n, 1), and $n-1$ slabs on either side having dimensions ($n-1$, n,1), ($n-2$, n, 1),.......(1, n,1), respectively, are cut off and on reaching the slabs of sides (1, n,1), on either end of the gnomon, the

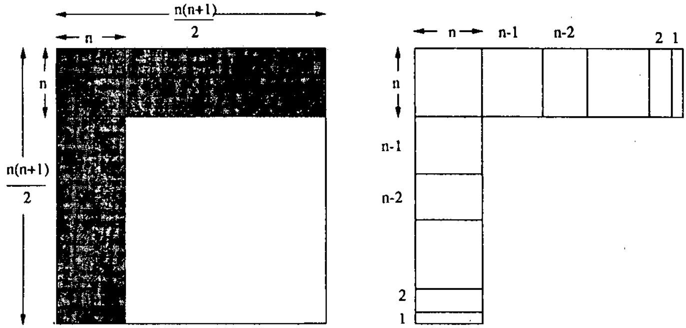

Fig. 10: A square slab of sides $n(n+1)/2$ and thickness 1. The gnomon which is to be cut off is indicated. Subdivision of the gnomon on either side is also indicated.

whole gnomon is found exhausted because the total length of all the slabs on each side of the gnomon is equal to

$$n+(n-1)+(n-2)+................+2+1 = \frac{n(n+1)}{2}$$

he side of the original square slab of sides $\frac{n(n+1)}{2}$ and of thickness 1

= the side of the gnomon cut off the original slab (see Fig.10).

Thus a square slab of side n with thickness 1 and two sets of $n-1$ rectangular slabs of thickness 1, one side n and the other side having values $n-1$, $n-2$,............,2,1, are obtained from the gnomon of width n sliced from the original square slab representing $(n(n+1)/2)^2$ From the two sets of $n-1$ rectangular slabs, the first retangular slab of sides ($n-1$, n, 1) from the first set is joined with the last slab of sides (1, n ,1) from the second set to get a square slab of sides n and thickness 1. Likewise, the second slab of sides ($n-2$, n,1) from the first set is joined with the last but one of sides (2, n ,1) from the second set to get a square slab of sides n and thickness 1. All the slabs from the two sets are exhausted in this manner by joining a slab of the first set from one end with a slab of the second set from the other end in the above said manner, and $n-1$ square slabs of side n and thickness 1 are built. These $n-1$ square slabs of side n and thickness 1 are joined and then joined together with the corner square slab of side n and thickness 1 to get a square block of side n with thickness n which is just a cube of side n (see Fig.11).

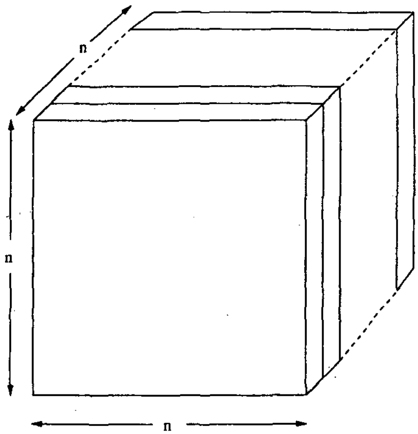

Fig. 11: Cube of side n constructed from the gnomon.

Thus a cube representing n^3 is formed from the gnomon of width n cut off from the original slab representing $(n(n+1)/2)^2$. Now from the remaining portion of the original slab, a gnomon of width $n-1$ is cut off and subdivided into successively decreasing smaller slabs which are then joined after rearranging as before to form a cube of side $n-1$, thus generating $(n-1)^3$ from the gnomon of width $n-1$. Proceeding like this, $(n-2)^3$, $(n-3)^3$,........, 2^3, 1^3 are generated respectively from the gnomons of width $n-2$, $n-3$, ...,2, 1 cut off from the remaining portions of the slab and the original slab of side $n(n+1)/2$ finally gets exhausted with the construction of the cube representing 1^3. Thus cubes of sides n, $n-1$, $n-2$,......,2,1 ,are built up from the original slab of sides $n(n+1)/2$ and thickness 1 and so the totality of all these cubes is the original block. Thus we have geometrically proved the relation

$$1^3+2^3+3^3+\ldots\ldots\ldots\ldots+n^3=\left(\frac{n(n+1)}{2}\right)^2.$$

Conclusion

Thus we find that Nilakantha has given excellent geometrical demonstrations for the sums of certain series of natural numbers. His disciple Sankara Varier of Trikuttaveli has included all these demonstrations in his commentary *Kriyakramakari* on the *Lilavati* of

Bhaskara II, while commenting on Bhaskara's enunciations on the sums of natural numbers. In addition, Sankara extends the formulae for the sum of sums of natural numbers to higher order repetitions and indicates that the formula for repeated sums beyond two can also be established in a similar manner. But he has not ventured to include the proof of this in the commentary, because it will not be easy for all to conceive or visualise figures in space having more than three dimensions. For demonstrating formula for sum of sums, three dimensional figures have been constructed and for establishing the formula for repeated sums higher dimensional figures may have to be visualised which will be difficult for the less intelligent.

These fine visual demonstrations exhibit the geometrical acumen of Nilakantha with which he imparted knowledge to his disciples like Sankara. Such methods are found to have been used by *acaryas* in those days for imparting knowledge to their disciples in the most convincing manner. Some of these are found to have been preserved for the use of later generations through the works like *Aryabhatiyabhashya* of Nilakantha, *Kriyakramakari* of Sankara and Narayana etc. Such visual demonstrations have the merit of making the abstract concepts under discussion at once convincing for the user. Such devices can be included in the present mathematical curriculum to introduce various mathematical concepts so that they are easily understood and visualised by the receiver. Thus a study of our ancient and medieval literature not only unfolds the richness of our tradition in mathematics, but also helps in framing an excellent curriculum in mathematics.

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Derivation of the Samskaras applied to the Madhava Series in Yuktibhasha

Jolly K. John

Accurate values of π in Kerala mathematics

The medieval mathematicians of the Kerala School were highly successful in the determination of the circumference of a circle with a given diameter, which can be translated into impressively accurate values for π. The value of π can be obtained correct to several decimal places from the various texts of the Kerala School.

Nilakantha (1444-1545 AD) in his *Tantrasangraha* (1500 AD) says, "Multiply any given diameter by 1,04,348 and divide the product by 33,215; the quotient is a very accurate circumference". This is the same as taking $\pi = 3.141592653921$, which is an approximation correct to 9 decimal places. *Karanapaddhati* (1732 AD), and *Sadratna-mala* (1823 AD) state that, "if the circumference of a circle in minutes be multiplied by 10,000,000,000 (10^{10}) and the product be divided by 3,141,59,26,536, the quotient will be the diameter of the circle in terms of the minutes of the circumference, and its half will be the radius". The value of π is correct to 10 decimal places here.

Kriyakramakari (1550 AD) quotes Madhava and says, "for a diameter of 900,000,000,000 (9×10^{11}) the circumference is 28,27,43,33,88,233". The value of π from this is 3.14159265359 which is correct to 11 decimal places. *Sadratnamala* (1823 AD) says "if you measure the circumference of a great circle by 100,000,000,000,000,000 (i.e., 10^{17}) parts, the circumference will be equal to 314159265358979324 of such parts", which gives π correct to 17 decimal places.

Charles M. Whish, one of the few Europeans to recognize the medieval Indian contributions to mathematics, states in his paper of 1835:

> The approximations to the true value of the circumference with a given diameter, exhibited in these three works (*Tantrasangraha, Karanapaddhati,* and *Sadratnamala*) are so wonderfully correct, that European mathematicians who seek for such proportion in the doctrine of fluxions, or in the more tedious continual bisection of an arc, will wonder by what means the Hindu has been able to extend the proportion to so great a length.

The question raised by Whish has only partly been answered till now. But in the last 50 years there have been some important attempts to understand the method by which the Kerala mathematicians arrived at such wonderfully accurate values for π.

Madhava's scheme for the computation of π

It is now accepted that the scheme for determining the circumference of a circle with a given diameter was originally given by the great Kerala astronomer-mathematician Madhava of Sangamagrama (1340-1425 AD). The scheme has essentially two parts. In the first part, the Madhava series for the circumference (C) of a circle with a given diameter(d) is obtained as,

$$C = 4d - \frac{4d}{3} + \frac{4d}{5} - \frac{4d}{7} + \text{------------} \qquad (1)$$

The derivation of this series involves clever geometrical constructions. and application of infinitesimal arguments and limits of series. This derivation itself shows the heights of mathematical ingenuity the Kerala School had developed.

The second part of Madhava's scheme consists of introducing a *samskara* or a correction term to the Madhava series (1), after considering only a finite number of terms of the infinite series. The necessity of introducing the correction arises from the fact that the series (1) converges very slowly, and for any practical computation, one has to stop at a finite number of terms. The correction term takes care of the contribution from the remaining terms of the series. In *Yuktidipika,* a commentary on *Tantrasangraha,* we see the Madhava series with the following two forms of corrections.

$$C = 4d\left(1 - \frac{1}{3} + \frac{1}{5} - \ldots\ldots\ldots \pm \frac{1}{p} \mp \frac{\frac{(p+1)}{2}}{(p+1)^2+1}\right), \qquad (2)$$

$$C = 4d\left(1 - \frac{1}{3} + \frac{1}{5} - \ldots\ldots \pm \frac{1}{p} \mp \frac{\left(\frac{p+1}{2}\right)^2+1}{\left[(p+1)^2+4+1\right]\left(\frac{p+1}{2}\right)}\right) \qquad (3)$$

Here p is the last odd number at which the series is truncated and is followed by the correction term. The correction term, called *samskara* ensures a far more correct value for the circumference than the value obtained without the correction term.

The reconstruction and the study of the method of arriving at the Madhava series has been done satisfactorily by many authors. But, so far there has not been a convincing explanation for the method to obtain the correction terms. There have been two major attempts in the past for the reconstruction of the historical derivation of the correction terms; one by Youschkevitch (1964 AD) and another by Hayashi, Kusuba, and Yano (1990 AD).

Youschkevitch conjectures that Madhava started with the value $\pi = \frac{377}{120}$ and then obtained the correction terms by considering successive terms of the Madhava series. Hayashi *et al.* first attempt a reconstruction of the derivation of the correction based on *Kriyakramakari* and propose a hypothesis which states that Madhava must have started with the value $\pi = \frac{355}{113}$, and then arrived at the correction terms and subsequently tried to give a deductive proof.

Here we reconstruct the derivation of the correction terms based on the celebrated Malayalam text *Yuktibhasha*. *Yuktibhasha* is a unique text among the many produced by the Kerala mathematicians and astronomers for two reasons: (i) it is written in Malayalam while most of the other texts are written in Sanskrit, (ii) it is a text exclusively written for explaining the rationale of the many mathematical results developed/used by the Kerala School. The rationale given in *Yuktibhasha* is sufficient for the reconstruction of the entire process of arriving at the correction terms in Eqs. (2) and (3). The leads provided by Rama Varma (Maru) Thampuran and A.R.Akhileshwara Aiyer in their edition of *Yuktibhasha* have been extremely useful here.

Yuktibhasha's derivation of the correction terms

Yuktibhasha first gives the derivation of Eq.(1), *i.e.*, the Madhava series without the correction term, and then proceeds to derive the correction terms with the following explanation:

> Here it is explained how doing a correction to the results of division by odd numbers above and above, will give (a result) closer to the circumference. At first it is to be determined whether this correction is precise or not. For

> that, obtain the result after division by a certain odd number and do the correction. Then (seperately) obtain the result after division by the next odd number and do the correction using the even number above this. If the (two) circumferences thus obtained are equal, then the corrections are precise.

The series for the circumference is the result of division by successive odd numbers. Let the series be terminated with a certain odd number say $(p-2)$, and then a correction be applied. Let the denominator of the correction term be S_1. Then the circumference is:

$$C = 4d\left[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\text{-----}\quad -\frac{1}{(p-2)}+\frac{1}{S_1}\right]. \tag{4}$$

Next, instead of stop-computing the series at the odd number $(p-2)$, include the term for the next odd number p also, and then write the series for the circumference. Then a different correction term has to be introduced. Let S_2 be the denominator of the new correction term. The new expression for the circumference is:

$$C = 4d\left[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\text{-----}\quad -\frac{1}{(p-2)}+\frac{1}{p}-\frac{1}{S_2}\right]. \tag{5}$$

If the two expressions obtained for the circumferences are numerically equal, then the corrections are exact. As *Yuktibhasha* notes:

> If the difference between the result of division by the odd number above and the correction term is equal to previous correction, then only the circumferences are equal. Then the sum of the two corrections will be equal to the result of division by the odd number. Corrections should be done in such a manner that the above equality occurs.

If the two circumferences are equal then the *RHS* of Eqs. (4) and (5) can be equated. This gives:

$$\frac{1}{S_1}=\frac{1}{p}-\frac{1}{S_2} \text{ or } \frac{1}{S_1}+\frac{1}{S_2}=\frac{1}{p} \tag{6}$$

It is clear that the corrections must satisfy this equality if they are to be exact.

It is obvious that $S_1 = S_2 = 2p$ satisfies Eq.(6). But both the corrections should follow the same rule. If we start with $S_1 = 2p$, then the rule is that the correction-divisor is double the odd number above the last odd number in the series (which is $p-2$). Then the next correction should be $2(p+2) = 2p+4$. On the other hand, if we start with $S_2 = 2p$,

then the rule is that the correction-divisor is double the last odd number in the series. Then one should take $S_1 = 2(p-2) = 2p-4$. In both the cases there is a difference of 4 between S_1 and S_2. Thus the possibility $S_1 = S_2 = 2p$ is ruled out. *Yuktibhasha* proceeds thus :

> Then what can be done is to take the two correction-divisors close to double the odd number concerned. If two numbers which differ by 2 are doubled, then the difference will be 4. The same difference will persist if some number is added to or subtracted from them and then doubled. Therefore one divisor is twice the odd number minus 2 and the other is twice the odd number plus 2. In order to get this, it is said that the correction-divisor is twice the even number above.

Since the exact solution of Eq.(6), $S_1 = S_2 = 2p$ is ruled out, the next best option is to have S_1 and S_2 close to $2p$. While writing down the corrections we have to make sure that both the corrections should follow the same rule. Since the correction-divisors we are looking for are functions of double the last odd number, a difference of 4 should be maintained between them. All these conditions are satisfied if $S_1 = 2p - 2$ and $S_2 = 2p + 2$. The rule here is that the correction-divisor is double the even number above the last odd number of the series.

Last odd number	Even number above	Correction-Divisor
$p-2$	$p-1$	$2(p-1) = 2p-2$
p	$p+1$	$2(p+1) = 2p+2$

As *Yuktibhasha* notes :

> Then in order to know the error (*sthaulya*), calculate the difference between the sum of the corrections and the result of division by the odd number inbetween. This is possible if the two correction-divisors and the odd number are converted to a common denominator..... When double the even number above is taken as the correction-divisor, the error obtained is 4 times the diameter divided by cube of the last odd number minus the root. If this is the case the correction is more than necessary.

The crucial step here which helps one to proceed with the derivation of the correction term is the computation of *sthaulya* or error. The error is defined as,

$$E(p) = \frac{1}{S_1} + \frac{1}{S_2} - \frac{1}{p} \qquad (7)$$

With the suggested values for S_1 and S_2, the error is computed now.

$$E(p) = \frac{1}{2p-2} + \frac{1}{2p+2} - \frac{1}{p} = \frac{4}{4p^3-4p}. \quad (8)$$

The error here is positive which indicates that the correction is more than necessary. Therefore the correction-divisor has to be increased so that the correction itself is reduced. As *Yuktibhasha* notes:

> Then to know how to reduce the corrections, suppose 1 is added to both the divisors.

Let 1 be added to the correction-divisor, so that the magnitude of the correction itself is reduced. Then, $S_1 = 2p-1$ and $S_2 = 2p+3$, Computing the error again,

$$E(p) = \frac{1}{2p-1} + \frac{1}{2p+3} - \frac{1}{p} = \frac{-2p+3}{4p^3+4p^2-3p}. \quad (9)$$

Compare this error with the error obtained in the previous step, *i.e.*, of (8). For sufficiently large values of p, the changes in the denominator can be neglected. Looking at the numerator we see that in Eq.(9) the error has increased to the variable's place while it was only in the unit's place in Eq.(8). Therefore the magnitude of the error has increased in Eq.(9). By adding and subtracting 1, we get further values of S_1 and S_2. We tabulate the corresponding errors $E(p)$ in Table 1.

No.	$\frac{1}{S_1}$	$\frac{1}{S_2}$	$E(p) = \frac{1}{S_1} + \frac{1}{S_2} - \frac{1}{p}$
1.	$\frac{1}{2p-5}$	$\frac{1}{2p-1}$	$\frac{6p-5}{4p^3-12p^2+5p}$
2.	$\frac{1}{2p-4}$	$\frac{1}{2p}$	$\frac{4p}{4p^3-8p^2}$
3.	$\frac{1}{2p-3}$	$\frac{1}{2p+1}$	$\frac{2p+3}{4p^3-8p^2-3p}$
4.	$\frac{1}{2p-2}$	$\frac{1}{2p+2}$	$\frac{4}{4p^3-4p}$

5.	$\frac{1}{2p-1}$	$\frac{1}{2p+3}$	$\frac{-2p+3}{4p^3+4p^2-3p}$
6.	$\frac{1}{2p}$	$\frac{1}{2p+4}$	$\frac{-4p}{4p^3+8p^2}$
7.	$\frac{1}{2p+1}$	$\frac{1}{2p+5}$	$\frac{-6p-5}{4p^3+12p^2+5p}$

Table 1: First order corrections

For large values of p, the changes in the denominator can be neglected and it can be taken to be approximately equal to $4p^3$ in all the cases. For $S_1 = 2p-2$ and $S_2 = 2p+2$, the numerator has only unit's place. For all the cases on either side of this, the numerator has the variable's place also. Therefore, the magnitude of the error is minimum for $S_1 = 2p-2$ and $S_2 = 2p+2$. Increasing or decreasing the correction-divisors by unity increases the magnitude of the error. Thus we conclude that the first order correction for a series ending with the odd number p is $\frac{1}{2p+2}$, *i.e.*,

$$F_1(p) = \frac{1}{2p+2} \tag{10}$$

Now, this correction is giving a finite error. Decreasing the correction-divisor by unity gives an error of the same sign but increased magnitude. Increasing the correction-divisor by unity gives an error, of increased magnitude and opposite sign. This shows that in order to achieve zero error the correction-divisor has to be increased by a quantity whose magnitude is less than unity. As *Yuktibhasha* notes:

> Then, 4 units divided by itself should be added to the correction-divisors......Declaring that (the result is) now almost precise, the *acharya* has directed to add 4 units divided by itself.

The way to obtain a better correction is to add fractions of the correction-divisor itself to the correction-divisor $2p+2$ in Eq.(10). The correction then takes the form.

$$F_2'(p) = \cfrac{1}{2p+2 + \cfrac{A}{2p+2}}$$

The error is now computed for various values of A (see Table 2) in an attempt to determine the value of A corresponding to the minimum error.

For sufficiently large values of p, the denominator of all the errors computed can be taken to be the same, i.e, $16p^5$. Then looking at the numerators, it becomes clear that $A = 4$ corresponds to the minimum error. Above and below this value, the magnitude of the error increases. The second order correction can then be defined as,

$$F_2(p) = \cfrac{1}{2p+2+\cfrac{4}{2p+2}} \tag{11}$$

This can be rewritten as,

$$F_2(p) = \frac{2p+2}{(2p+2)^2+4} = \frac{\frac{p+1}{2}}{(p+1)^2+1}.$$

This is the correction of Eq.(2).

A	$E(p)=\frac{1}{S_1}+\frac{1}{S_2}-\frac{1}{p}$
1	$\frac{12p^2-25}{16p^5-24p^3+25p}$
2	$\frac{8p^2-36}{16p^5-16p^3+36p}$
3	$\frac{4p^2-49}{16p^5-8p^3+49p}$
4	$\frac{-64}{16p^5+64p}$
5	$\frac{-4p^2-81}{16p^5+8p^3+81p}$
6	$\frac{-8p^2-100}{16p^5+16p^3+100p}$

Table 2: Second order corrections.

The third order correction can be derived by following the same procedure as employed to derive the correction of Eq.(11). Examination of Table 2 reveals that though the error for $A = 4$ is minimum, it has a finite value and is negative. Since the error is computed from Eq.(7), a negative value indicates that the correction is less than necessary. In order to increase the magnitude of the correction, its denominator is decreased by extending the continued fraction to one more term. A correction of the following form is assumed,

$$F_3'(p) = \cfrac{1}{2p+2+\cfrac{4}{2p+2+\cfrac{B}{2p+2}}}$$

With this kind of a correction, the error is now computed for various values of B. Then the value of B for which the minimum error obtained is selected. The results are tabulated in Table 3.

B	$E(p) = \frac{1}{S_1} + \frac{1}{S_2} - \frac{1}{p}$
12	$\frac{-64p^2+1600}{64p^7+320p^5+1216p^3-1600p}$
13	$\frac{-48p^2+1764}{64p^7+352p^5+1348p^3-1764p}$
14	$\frac{-32p^2+1936}{64p^7+384p^5+1488p^3-1936p}$
15	$\frac{-16p^2+2116}{64p^7+416p^5+1636p^3-2116p}$
16	$\frac{2304}{64p^7+448p^5+1792p^3-2304p}$
17	$\frac{16p^2+2500}{64p^7+480p^5+1956p^3-2500p}$

18	$\dfrac{32p^2+2704}{64p^7+512p^5+2128p^3-2704p}$
19	$\dfrac{48p^2+2916}{64p^7+544p^5+2308p^3-2916p}$
20	$\dfrac{64p^2+3136}{64p^7+576p^5+2496p^3-3136p}$

Table 3: Third order corrections.

Examination of the table makes it clear that $B = 16$ corresponds to the minimum error. On both sides of this value the magnitude of the error increases. The third order correction can then be written as,

$$F_3(p) = \cfrac{1}{2p+2+\cfrac{4}{2p+2+\cfrac{16}{2p+2}}} \tag{12}$$

This may be rewritten as,

$$F_3\,(p) = \frac{1}{2p+2+\dfrac{4(2p+2)}{(2p+2)^2+16}}$$

$$= \frac{(2p+2)^2+16}{(2p+2)^3+16(2p+2)+4(2p+2)}$$

$$= \frac{\left(\dfrac{p+1}{2}\right)^2+1}{\left[(p+1)^2+4+1\right]\left(\dfrac{p+1}{2}\right)}.$$

This expression is the same as the correction of Eq.(3).

Thus we see that the two corrections given in *Yuktidipika* are in fact continued fraction expressions which can be derived through an error minimisation algorithm given in *Yuktibhasha*. Now the question is whether the error can further be reduced and better corrections be found? The algorithm which is used to find the first three corrections is iterative in

nature and therefore can be carried out further to find a new correction at each step which is better than the previous one.

Let us compute the fourth order correction by carrying out the process once again. The correction we are looking for is of the form,

$$F_4'(p) = \cfrac{1}{2p+2+\cfrac{4}{2p+2+\cfrac{16}{2p+2+\cfrac{C}{2p+2}}}}$$

The error is now computed for various values of C. The results are tabulated in Table 4.

Examination of the table of errors reveals that $C = 36$ corresponds to the minimum possible error at this stage.

C	$E(p) = \frac{1}{S_1} + \frac{1}{S_2} - \frac{1}{p}$
30	$\frac{1536p^2 - 112895}{256p^9 + 5376p^7 + 38986p^5 - 16384p^3 + 112895p}$
31	$\frac{1280p^2 - 118336}{256p^9 + 5504p^7 + 40592p^5 - 11384p^3 + 118336p}$
32	$\frac{1024p^2 - 123904}{256p^9 + 5632p^7 + 42240p^5 - 16384p^3 + 123904p}$
33	$\frac{768p^2 - 129600}{256p^9 + 5760p^7 + 43920p^5 - 16384p^3 + 129600p}$
34	$\frac{512p^2 - 135424}{256p^9 + 5888p^7 + 45632p^5 - 16384p^3 + 135424p}$
35	$\frac{256p^2 - 141376}{256p^9 + 6016P^7 + 47386p^5 - 16384p^3 + 141376p}$
36	$\frac{-147456}{256p^9 + 6144p^7 + 49152p^5 - 16384p^3 + 147456p}$

37	$\dfrac{-256p^2-153664}{256p^9+6272p^7+50960p^5-16384p^3+153664p}$
38	$\dfrac{-512p^2-160000}{256p^9+6400p^7+52800p^5-16384p^3+160000p}$
39	$\dfrac{-768p^2-166464}{256p^9+6528p^7+54672p^5-16384p^3+166464p}$
40	$\dfrac{-1024p^2-173056}{256p^9+6656p^7+56576p^5-16384p^3+173056p}$
41	$\dfrac{-1280p^2-179776}{256p^9+6784p^7+58512p^5-16384p^3+179776p}$

Table 4: Fourth order corrections.

Thus we can write the 4th order correction as

$$F_4(p) = \cfrac{1}{2p+2+\cfrac{4}{2p+2+\cfrac{16}{2p+2+\cfrac{36}{2p+2}}}}. \tag{13}$$

No.	Correction	π	Accuracy
1.	No correction	3.12159 46525 91010	1
2.	$F_1(p) = \dfrac{1}{2p+2}$	3.14159 46525 91010	5
3.	$F_2(p) = \cfrac{1}{2p+2+\cfrac{4}{2p+2}}$	3.14159 26527 90990	8
4.	$F_3(p) = \cfrac{1}{2p+2+\cfrac{4}{2p+2+\cfrac{16}{2p+2}}}$	3.14159 26535 90510	10

5.	$F_4(p) = \dfrac{1}{2p+2+\dfrac{4}{2p+2+X}}$,	$\underline{3.14159}\ \underline{26535}\ \underline{89792}$	14

$$X = \frac{16}{2p+2+\dfrac{36}{2p+2}}$$

Table 5: Successive Approximations when p = 99. The last column (accuracy) specifies the number of decimal places which are correct.

In order to see the validity of this newly derived correction term we apply this to the Madhava series and compute the value of π. We may write,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ------ \pm \frac{1}{p} \mp F(p). \qquad (14)$$

We take $p = 99$ and all the four corrections are tried for $F(p)$, one by one. Table 5 gives the values of π obtained in each case.

We see that the corrections $F_2(p)$ and $F_3(p)$, which are mentioned in and derived from the rationale given in *Yuktibhasha,* dramatically improve the accuracy of π over the value obtained by using the Madhava series without the corrections. It is also evident that the newly derived correction $F_4(p)$ further improves the accuracy of π. This demonstrates the validity of the new correction term.

From the form of the different correction terms so far obtained, we conjecture that the next (i.e., 5th order) correction should be

$$F_5(p) = \cfrac{1}{2p+2+\cfrac{4}{2p+2+\cfrac{16}{2p+2+\cfrac{36}{2p+2+\cfrac{64}{2p+2}}}}}.$$

In order to check the validity of this correction, we compute π using this for $p = 99$. The value of π obtained is :

$$\pi = \underline{3.14159\ 26535\ 89793}\ 24130\ .$$

This is a value which is correct in the first 17 decimal places. This is the same as the value obtained by Raja Sankara Varma in his *Sadratnamala.*

Conclusion

1. The method of obtaining the corrections has a definite pattern. It is not just a trial and error method. It is an iterative algorithm in which one starts with a cleverly guessed initial value and then proceeds to find better and better approximations by error minimisation. The Madhava algorithm is similar to procedures used in computational mathematics today.
2. It appears that the incommensurability of the ratio $\frac{C}{d}$ is the underlying theme of the Madhava algorithm. The impossibility of obtaining exact values for the correction term is accepted at the outset. The attempt is only to find better and better approximations.
3. *Yukthibhasha* explains the derivation of first three orders of the correction. But since the algorithm is iterative in nature, one can proceed to derive the higher order corrections.
4. Since obtaining the higher order corrections is only a simple iteration of the algorithm, it is obvious that Madhava and his followers in the Kerala School could have carried out the necessary calculations without any difficulty. We may conclude that the medieval Kerala mathematicians could have calculated π with any desired accuracy if they wished so. The reason why they did not compute beyond a few decimal places may be that they did not have any 'use' for such finer values.

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Citation presented to Prof. K.V. Sarma

श्रीः

श्रीकुण्डग्रामज-गार्ग्यगोत्रोत्पन्न-नीलकण्ठसोमयाजिकृतानेकग्रन्थेषु बहुविश्रुततन्त्रसंग्रहाख्य
ग्रन्थस्य पञ्चशताब्दपूर्तौ चेन्नै-विश्वविद्यालयेन आयोजितसंमेलने
के.वी.शर्मेति विख्यात-विद्वत्तल्लजानां
श्रीकृष्णवेंकटेश्वरशर्मणां
विशिष्य केरलज्योतिर्विदां ग्रन्थप्रकाशने कृतभूरिपरिश्रमाणां
करकमलयोः समर्पितेयमभिनन्दनपत्रिका।।

भास्कराचार्य एवासीदन्तिमो गणकोत्तमः। इत्येवं दुष्प्रथा कैश्चित् पुस्तकेषु प्रकाशिता।।
भारतीयेषु शास्त्रेषु गणिते ज्योतिषे तथा। तर्कयुक्त्युपपत्तीनामभावं ते हि मेनिरे।।
प्रथायास्तथ्यतां ज्ञातुं ग्रन्थानां परिशीलनम्। कर्तव्यमुपलब्धिस्तु तेषां नासीद्यदा तदा।।

'केवीशर्म' महोदयैर्गणितधीसंयुक्तमेधाविनां
ग्रन्थानामुपलब्धये प्रयतनं सुश्लाघ्यमायोजितम्।
ग्रन्थान् सम्यगधीत्य पण्डितवरास्स्वान्तं मुदा पूरयन्
शास्त्रं नूनमपूर्वयुक्तिसहितं प्राकाश्यमापादयन्।।

वाक्यं वररुचेश्चैव तथा सिद्धान्तदर्पणम्। तन्त्रसंग्रह इत्याख्यग्रन्थयुक्तेस्तु दीपिका।।
नीलकण्ठकृतज्योतिर्मीमांसा गोलदीपिका। सूर्यय्ज्वकृतं भाष्यं क्रियाक्रमकरी तथा।।
वेण्वारोहो युक्तिभाषा गोलसारस्तथैव च। स्फुटचन्द्राप्तिरित्येवं ग्रन्था नैके प्रकाशिताः।।
अधीत्येमं ग्रन्थराशिं भवद्भिः संप्रकाशितम्। सा प्रथा नैव तथ्येति स्फुटं नः शेमुषीजुषाम्।।

यदा लुप्तप्राया गणितकृतयः केरलभुवां त्वया नूनं तासां कृतमकृतपूर्वं प्रकशनम्।
त्वदीयं कर्मेदं सहृदयबुधाह्लादनकरं ततोऽहं मन्ये त्वं सफलजनिरस्मिन् बुधकुले।।

शुष्यतो वृक्षजातस्य यथा शास्त्रस्य सेचनम्। कृत्वा प्रकाशनं तेन जीवितं प्रापितं पुनः।।
भवता चिरकालेन कृतो भूरिपरिश्रमः। यो हि तं सफलीकर्तुं कृषिं कुर्मो यथोचिताम्।।

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